A partial order on partial isometries (symmetric operators)
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Deficiency indices of $V$: $n_+ := \dim \ker(V)$, and $n_- := \dim \operatorname{ran}(V)^\perp$. 
Partial Isometries

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Deficiency indices of $V$: $n_+ := \dim \ker(V)$, and $n_- := \dim \text{Ran}(V)^\perp$.

We study c.n.u. (simple) partial isometries $V$ with equal deficiency indices:

$$n_+ = n = n_-.$$
Two Questions:

1. Given a unitary \( U \) on \( K \supset H \) when is \( U \) an extension of \( V \)?

A natural partial order on partial isometries:

\[ V_1 \in B(H_1), \quad V_2 \in B(H_2) \]

\( H_1 \subset H_2 \) if \( V_1 \subset V_2 \) if 

\[ \text{Ker}(V_1) \perp \subset \text{Ker}(V_2) \perp \]

and \( V_2 \mid \text{Ker}(V_1) \perp = V_1 \mid \text{Ker}(V_1) \perp \).

\( V_1 \preceq V_2 \) if \( V_1 \simeq V_1' \subset V_2 \), and \( \simeq \) denotes unitary equivalence.

This is a partial order on unitary equivalence classes.

A second partial order:

\( V_1 \preceq_{qs} V_2 \) if \( V_1 \) is quasi-similar to a partial isometry \( V_1' \subset V_2 \).
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$$V_1 \subset V_2 \text{ if } \ker(v_1)^\perp \subset \ker(v_2)^\perp \text{ and } V_2|_{\ker(v_1)^\perp} = V_1|_{\ker(v_1)^\perp}.$$
Unitary extensions / a natural partial order

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   - $V_1 \subset V_2$ if $\ker(V_1)^\perp \subset \ker(V_2)^\perp$ and $V_2|_{\ker(V_1)^\perp} = V_1|_{\ker(V_1)^\perp}$.

   - $V_1 \precsim V_2$ if $V_1 \simeq V'_1 \subset V_2$, and $\simeq$ denotes unitary equivalence.
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   - A natural partial order on partial isometries:
     
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     $\mathbf{V}_1 \subset V_2$ if $\ker(V_1) \perp \subset \ker(V_2) \perp$ and $V_2|_{\ker(V_1) \perp} = V_1|_{\ker(V_1) \perp}$.

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     $V_1 \subset V_2$ if $\text{Ker}(V_1) \perp \subset \text{Ker}(V_2) \perp$ and $V_2|_{\text{Ker}(V_1) \perp} = V_1|_{\text{Ker}(V_1) \perp}$.

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     This is a partial order on unitary equivalence classes.

   - A second partial order: $V_1 \preccurlyeq_{qs} V_2$ if $V_1$ is quasi-similar to a partial isometry $V_1' \subset V_2$. 
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   - This is a partial order on unitary equivalence classes.

   A second partial order: $V_1 \lesssim_{qs} V_2$ if $V_1$ is quasi-similar to a partial isometry $V_1' \subset V_2$.

2. Given partial isometries $V_i \in B(\mathcal{H}_i)$ when is $V_1 \lesssim V_2$? ($V_1 \lesssim_{qs} V_2$?)
The Cayley Transform

- There is a bijection between partial isometries and symmetric linear transformations:
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\begin{align*}
U \text{ unitary} & \iff b^{-1}(U) \text{ self-adjoint}, \\
A \text{ self-adjoint} & \iff b(A) \text{ unitary}.
\end{align*}
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- There is a bijection between partial isometries and symmetric linear transformations:

\[ b^{-1}(z) = i \frac{1+z}{1-z} \]
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- \( U \) unitary \( \iff \) \( b^{-1}(U) \) self-adjoint,
- \( A \) self-adjoint \( \iff \) \( b(A) \) unitary.

- \( V \) partial isometry \( \iff \) \( B := b^{-1}(U) \) symmetric \( (B \subset B^*) \),
- \( B \) symmetric \( \iff \) \( b(A) \) partial isometry.

- In general \( B \neq B^* \) (\( B \) is not self-adjoint!):

\[ \text{Ker}(B^* - i) = \text{Ker}(V) \quad \text{and} \quad \text{Ker}(B^* + i) = \text{Ran}(V)^\perp. \]
Examples: deBranges-Rovnyak spaces

\( H^2, H^\infty \) Hardy spaces on \( \mathbb{C}_+ \)
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$H^2, H^\infty$ Hardy spaces on $\mathbb{C}_+$

- $\Theta :=$ extreme point of the unit ball of $H^\infty$, and $K_\Theta^2 :=$ the deBranges-Rovnyak space.
  - e.g. If $\Theta$ is inner then $K_\Theta^2 = H^2 \ominus \Theta H^2$ is a model subspace.
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$H^2, H^\infty$ Hardy spaces on $\mathbb{C}_+$

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  - e.g. If $\Theta$ is inner then $K^2_\Theta = H^2 \oplus \Theta H^2$ is a model subspace.
- Let $S_i(\Theta) := \{ f \in K^2_\Theta \mid f(i) = 0 \}$.
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- $\Theta$ := extreme point of the unit ball of $H^\infty$, and $K^2_\Theta$ := the deBranges-Rovnyak space.
  - e.g. If $\Theta$ is inner then $K^2_\Theta = H^2 \ominus \Theta H^2$ is a model subspace.
- Let $S_i(\Theta) := \{ f \in K^2_\Theta | f(i) = 0 \}$.
- If $S :=$ multiplication by $b(z) = \frac{z-i}{z+i}$ (the shift) then $V^*_\Theta := S^*|_{S_i(\Theta)}$ is a partial isometry with indices $(1, 1)$.
- $Z_\Theta := b^{-1}(V_\Theta)$ acts as multiplication by $z$. 
deBranges-Rovnyak space examples II

\[ Z_\Theta := b^{-1}(V_\Theta) \]

We say \( \Theta_1 \leq \Theta_2 \) if \( \Theta_1^{-1}\Theta_2 \) is in the unit ball of \( H^\infty \).
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- If $\Theta_1 \leq \Theta_2$ are extreme but not inner then $Z_{\Theta_1} \precsim_{qs} Z_{\Theta_2}$:
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    $$EZ_{\Theta_1} = Z_{\Theta_2}' E \quad E^* Z_{\Theta_2}' = Z_{\Theta_1} E^*$$
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    \[ EZ_{\Theta_1} = Z'_{\Theta_1} E \quad E^* Z'_{\Theta_2} = Z_{\Theta_1} E^* \]
  - \( Z_{\Theta_1} \) is quasi-similar to \( Z'_{\Theta_2} \subset Z_{\Theta_2} \).
$S_n(\mathcal{H}) := \text{simple symmetric operators with indices } (n, n) \text{ acting in } \mathcal{H}$
Definition

If $B \in S_n(H)$, $\text{Ext}(B)$ := all self-adjoint $A$ with $\text{Dom}(A) \subset K \supset H$ such that $B \subset A$. $\text{Ext}_b(B)$ := all self-adjoint $A$ such that $B$ is quasi-similar to a restriction of $A$. 

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- For any $A \in \operatorname{Ext}_b(B)$ we construct a RKHS $\mathcal{H}_A$ of analytic functions on $\mathbb{C} \setminus \mathbb{R}$ and a co-isometry $U_A : \mathcal{H} \to \mathcal{H}_A$ such that:
  - $U_A B = Z_A U_A$
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  - $\mathcal{H}_A$ has reproducing kernel $k_w^A(z) := \Gamma(z)^* \Gamma(w)$ where

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    \]
  - $X$ is an injective linear map which implements a quasi-affine transform between $B$ and a restriction of $A$. 
Connection to complex function theory

Definition

Given \( B \in S_n(\mathcal{H}) \) the characteristic function, \( \Theta_B \) of \( B \) is the \( n \times n \) matrix-valued function on \( \mathbb{C}_+ \) defined by

\[
\Theta_B(z) := b(z) k_i(i) \frac{1}{2} k_i(z)^{-1} k_{-i}(z) k_{-i}(-i)^{-\frac{1}{2}}.
\]

Here \( k_w(z) = \Gamma_A(z)^* \Gamma_A(w) \) is the reproducing kernel for any model space \( \mathcal{H}_A \), \( A \in \text{Ext}_b(B) \).
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- $\Theta_B$ is contractive, analytic on $\mathbb{C}_+$. 
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- $\Theta_B$ is contractive, analytic on $\mathbb{C}_+$.
- $\Theta_B$ is independent of $A \in \text{Ext}_b(B)$ (up to conjugation by unitaries).
Connection to complex function theory

Theorem (Livsic, $n = \infty$ Aleman, M., Ross)

$B_j \in S_n$ are unitarily equivalent if and only if there are fixed unitary $U$, $V$ such that $\Theta_{B_1}(z) = U\Theta_{B_2}(z)V$. 
Connection to complex function theory

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Theorem

\( B_i \in S_n \) are quasi-similar \((n < \infty)\) if and only if there are fixed invertible \( R, Q \) such that 
\[ \Theta_{B_1}(z) = R\Theta_{B_2}(z)Q. \]
A larger RKHS

Recall that for $A \in \text{Ext}(B)$ $\mathcal{H}_A$ has kernel $k^A_w(z) = \Gamma_A(z)^*\Gamma_A(w)$ where

$$\Gamma_A(z) = X^*(A - i)(A - z)^{-1}XP_{\text{Ker}(B^* - i)}.$$
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- Define

  $$\Omega_A(z) := (A - i)(A - \bar{z})^{-1}XP_{\text{Ker}(B^* - i)},$$

  let $\mathcal{K}_A$ be the RKHS with kernel $K_w(z) := \Omega(z)^*\Omega(w)$. 
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  \]
  let $\mathcal{K}_A$ be the RKHS with kernel $K_w(z) := \Omega(z)^*\Omega(w)$.

- For $c > \|XX^*\|$, $cK_w(z) - k_w(z) = \Omega_A^*(z)(c - XX^*)\Omega_A(w)$
  is a positive kernel function so that $\mathcal{H}_A \subset \mathcal{K}_A$ boundedly by RKHS theory.
Fix $B \in S_1(\mathcal{H})$ and $A \in \text{Ext}_b(B)$
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- Recall: $k_w(z)$, $K_w(z)$ reproducing kernels for $\mathcal{H}_A \subset \mathcal{K}_A$ and
  $\Theta_B(z) := b(z)(k_i(i))^{1/2}(k_i(z))^{-1}k_{-i}(z)(k_{-i}(-i))^{-1/2}$
Characterization of $\text{Ext}(B)$

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**Theorem**

Let $\Lambda_A(z) := b(z)(K_i(i))^{1/2}(K_i(z))^{-1}K_{-i}(z)(K_{-i}(-i))^{-1/2}$, then $\Lambda_A \geq \Theta_B$ i.e. $\frac{\Lambda_A}{\Theta_B}$ is contractive.
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**Theorem**

There is a bijection $A \in \text{Ext}_b(B) \mapsto \Phi_A$ onto the set of all contractive analytic $\Phi$ such that $\frac{\Phi - \Phi(i)}{1 - \Phi(i)\Phi} = \Lambda \geq \Theta_B$.

(General $n$ versions in progress)
Partial Order

Theorem

If $B_1 \preceq_{qs} B_2$ belong to $S_1(H_i)$ with characteristic functions $\Theta_i = \Theta_{B_i}$ then there is a contractive analytic $\tilde{\Theta}_2$ with Herglotz measure absolutely continuous with respect to that of $\Theta_2$ such that $\tilde{\Lambda}_2 := \frac{\tilde{\Theta}_2 - \tilde{\Theta}_2(i)}{1 - \tilde{\Theta}_2(i)\tilde{\Theta}_2} \geq \Theta_1$.
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**Theorem**

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- This needs to be refined as the converse is not true in general.
Partial Order

**Theorem**

*If* $B_1 \lessapprox_{qs} B_2$ *belong to* $S_1(\mathcal{H}_i)$ *with characteristic functions* $\Theta_i = \Theta_{B_i}$ *then there is a contractive analytic* $\tilde{\Theta}_2$ *with Herglotz measure absolutely continuous with respect to that of* $\Theta_2$ *such that* $\tilde{\Lambda}_2 := \frac{\tilde{\Theta}_2 - \tilde{\Theta}_2(i)}{1 - \tilde{\Theta}_2(i)\tilde{\Theta}_2} \geq \Theta_1$.

- This needs to be refined as the converse is not true in general.

**Theorem**

*If* $B_1, B_2 \in S_n(\mathcal{H}_i)$, $n \in \mathbb{N} \cup \{\infty\}$ *with characteristic functions* $\Theta_i$, *and* $\Theta_1 \leq \Theta_2$ *then* $B_1 \lessapprox_{qs} B_2$.

- **Q:** What happens when $B_1 \in S_n(\mathcal{H}_1)$ and $B_2 \in S_m(\mathcal{H}_2)$ and $n \neq m$?
There are several interesting connections with complex function theory.
Outlook

- There are several interesting connections with complex function theory.
- A number of open questions remain:
  1. How do we compare symmetric operators (partial isometries) with different deficiency indices?
  2. Extension of results to arbitrary c.n.u. contractions with defect indices \((n, n)\)?
  3. Let \(POVM(B)\) be the convex set of all POVMs \(Q(\Omega) := X^* \chi(\Omega) A X\) for \(A \in \text{Ext}_b(B)\). (Naimark dilation \(\Rightarrow\) convex.) What are its extreme points?