The Choquet boundary of an operator system

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joint work with Matthew Kennedy
This is dedicated to the memory of William B. Arveson

Two recent surveys of Bill’s work in JOT:

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**Theorem (Sz.Nagy (1953))**

If \( T \in \mathcal{B}(\mathcal{H}) \) and \( \| T \| \leq 1 \), there is a unitary operator of form

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U = \begin{bmatrix}
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**Corollary (Generalized von Neumann Inequality)**

If $[p_{ij}]$ is a matrix of polynomials, and $\|T\| \leq 1$, then

$$\| [p_{ij}(T)] \| \leq \sup_{|z| \leq 1} \| [p_{ij}(z)] \|.$$
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Hence this can be considered as a study of representations of the disk algebra \( A(\mathbb{D}) \).
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- **Operator algebra** \( \mathcal{A} \): unital subalgebra of a C*-algebra \( \mathcal{C}^*(\mathcal{A}) \).
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- **The role of completely positive and completely bounded maps.**

$\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ induces

$$\varphi_n : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n)$$

by

$$\varphi_n([a_{ij}]) = [\varphi(a_{ij})].$$
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Say \( \varphi \) is **completely bounded** (c.b.) if

\[ \| \varphi \|_{cb} = \sup_{n \geq 1} \| \varphi_n \| < \infty. \]

Say \( \varphi \) is **completely contractive** (c.c.) if \( \| \varphi \|_{cb} \leq 1 \).
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Ken Davidson and Matt Kennedy  The Choquet boundary 5 / 22
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If $\rho : A \rightarrow B(\mathcal{H})$ is c.c., then $S = \overline{A + A^*}$ and

$$\tilde{\rho}(a + b^*) = \rho(a) + \rho(b)^*$$

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**Theorem (Arveson’s Extension Theorem)**

*If \( \varphi : S \to B(\mathcal{H}) \) is c.p. and \( S \subset T \), then there is a c.p. map \( \psi : T \to B(\mathcal{H}) \) s.t. \( \psi|_S = \varphi \). i.e. \( B(\mathcal{H}) \) is injective.*
A dilation of a c.c. representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a c.c. representation $\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ where $\mathcal{K} = \mathcal{K}_- \oplus \mathcal{H} \oplus \mathcal{K}_+$, and

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A dilation of a u.c.p. map $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is a u.c.p. map $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{K})$ where $\mathcal{K} = \mathcal{H} \oplus \mathcal{K}'$ and $P_{\mathcal{H}}\psi(a)|_{\mathcal{H}} = \varphi(a)$:

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Note that if $\sigma \succ \rho$, then $\tilde{\sigma} \succ \tilde{\rho}$.
But $\psi \succ \tilde{\rho}$ may not be multiplicative on $\mathcal{A}$. 

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Theorem (Arveson’s Dilation Theorem)

Let $\rho : A \to \mathcal{B}(\mathcal{H})$ be a representation. TFAE

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Now we turn to two central ideas in Arveson’s paper which he was not able to verify in general:

- boundary representations
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Bill was able to verify this in many concrete examples. See also Subalgebras of $C^*$-algebras II, Acta Math. **128** (1972), 271–308.
A u.c.p. map $\varphi : S \to B(H)$ or a c.c. repn. $\varphi : A \to B(H)$ has the unique extension property (u.e.p) if

1. $\varphi$ has a unique u.c.p. extension to $C^*(S)$ (or $C^*(A)$)
2. this extension is a $*$-homomorphism

It is a boundary representation if it has u.e.p. and

3. the $*$-homomorphism is irreducible.
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If $1 \in A \subset C(X)$, then irreducible repns. are point evaluations $\delta_x$. A u.c.p. extension is given by a measure $\mu$ on $X$ such that

$$f(x) = \int_X f \, d\mu \quad \text{for all} \quad f \in A.$$
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Thus \( \delta_x \) is a boundary representation

\[\iff\] \( x \) has a unique representing measure

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$\iff$ $x$ is in the Choquet boundary of $A$.

The boundary representations form the **Choquet boundary of $S$**.
The $C^*$-envelope of $\mathcal{A}$ is a pair $(C^\ast_{\text{env}}(\mathcal{A}), \iota)$ where $\iota: \mathcal{A} \to C^\ast_{\text{env}}(\mathcal{A})$ is comp. isom. iso., $C^\ast_{\text{env}}(\mathcal{A}) = C^\ast(\iota(\mathcal{A}))$, with universal property: if $j: \mathcal{A} \to \mathcal{B} = C^\ast(j(\mathcal{A}))$ comp. isom. iso. then $\exists q: \mathcal{B} \to C^\ast_{\text{env}}(\mathcal{A})$ $\ast$-homomorphism s.t. $q j = \iota$.

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\downarrow j & & \uparrow q \\
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If there are sufficiently many boundary representations $\{\pi_{\lambda}\}$
to completely norm $\mathcal{S}$, let $\pi = \bigoplus \pi_{\lambda}$. Then
\[ C^*_\text{env}(\mathcal{S}) = C^*(\pi(\mathcal{S})). \]
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**Theorem (Hamana (1979))**

Every operator system is contained in a unique minimal injective operator system.

**Corollary (Hamana)**

*Every operator system has a $C^*$-envelope.*
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**Muhly-Solel (1998)** gave a homological characterization of boundary representations.
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This dilation proof yields important information about $C^*_{\text{env}}(\mathcal{A})$.
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Muhly-Solel result says: a repn. has u.e.p. $\iff$ it is an extremal extension and an extremal coextension.
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- reworks Dritschel-McCullough for operator systems
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**Theorem (Arveson (JAMS 2008))**

If $S$ is separable, then there are sufficiently many boundary representations.
Our approach

- We give a dilation theory proof of the existence of boundary representations.
- It works in complete generality.
- The argument is conceptual and natural.
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Then $\psi(a) = P \varphi(a)$ satisfies $0 \leq \psi \leq \varphi$ but
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Arveson (2008) Say $\varphi$ is **maximal** at $(s, x)$ if
$$\psi \succ \varphi \implies \|\psi(s)x\| = \|\varphi(s)x\|.$$ 

If $\varphi$ is maximal at every $(s, x)$, then $\varphi$ is maximal.
**Key Lemma**

If $\varphi$ is pure, and $(s_0, x_0) \in S \times \mathcal{H}$, then there is a pure dilation $\psi : S \to B(\mathcal{H} \oplus \mathbb{C})$ s.t. $\psi \succ \varphi$ and $\psi$ is maximal at $(s_0, x_0)$.
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- If $\psi : S \to B(\mathcal{H} \oplus \mathcal{K})$, then compression to $\text{span}\{\mathcal{H}, \psi(s_0)x_0\}$ has same norm at $(s_0, x_0)$. 
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- \( \{\psi : S \to B(\mathcal{H} \oplus \mathbb{C}) : \psi \succ \varphi\} \) is BW-compact.
  
  Hence \( \exists \psi \) s.t. \( \psi(s_0)x_0 = \varphi(s_0)x_0 \oplus \eta \) with \( \eta \) maximal.
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  \( \{\psi : S \to B(\mathcal{H} \oplus \mathbb{C}) : \psi \succ \varphi, \ \psi(s_0)x_0 = \varphi(s_0)x_0 \oplus \eta\} \).
- Delicate argument to show that \( \psi_0 \) is pure.
Theorem 1

Every pure u.c.p. map $\varphi : S \to B(\mathcal{H})$ dilates to a maximal pure u.c.p. map, and hence extends to a boundary representation.
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- routine transfinite induction to obtain dilation maximal at every pair $(s, x)$
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Every pure u.c.p. map \( \varphi : S \to \mathcal{B}(\mathcal{H}) \) dilates to a maximal pure u.c.p. map, and hence extends to a boundary representation.

- routine transfinite induction to obtain dilation maximal at every pair \((s, x)\)

- if \( S \) is separable and \( \dim \mathcal{H} < \infty \), then can produce the maximal dilation as limit of sequence of finite dim. maps.
**Theorem 2**

*There are sufficiently many boundary representations to completely norm $S$.*
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In the next four decades, our approach 

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**Theorem 2**

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- Take \( S \in \mathcal{M}_n(S) \). Suffices to norm \( T = S^*S \).
- Choose pure state \( \varphi \) on \( \mathcal{M}_n(S) \) that norms \( T \).
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- Take $S \in M_n(S)$. Suffices to norm $T = S^*S$.
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- Dilate it to a boundary repn. $\sigma$ of $M_n(S)$ by Theorem 1. Then $\sigma \simeq \pi^{(n)}$, where $\pi$ is irreducible repn. of $C^*(S)$. 
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- Dilate it to a boundary repn. $\sigma$ of $\mathcal{M}_n(S)$ by Theorem 1. Then $\sigma \simeq \pi^{(n)}$, where $\pi$ is irreducible repn. of $C^*(S)$.
- If $\varphi$ is u.c.p. dilation of $\pi|_S$, then $\varphi^{(n)}$ dilates $\sigma|_{\mathcal{M}_n(S)}$. Hence $\varphi = \pi$. So $\pi$ is the desired boundary repn. (This is easy direction of a result of Hopenwasser.)
Second method to get sufficiently many boundary repns. A **matrix state** is a u.c.p. map of $S$ into $M_n$.

**Theorem**

The pure matrix states completely norm $S$. 

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**Theorem**
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- Finite dimensional compressions of a faithful repn. of $C^*(S)$ completely norm $S$. So matrix states completely norm $S$.
- The collection of all matrix states $(S_n(S))_{n \geq 1}$ is C*-convex: If $\gamma_j \in M_{n_j,n}$, $\sum_{j=1}^{k} \gamma_j^* \gamma_j = I_n$ and $\psi_j \in S_{n_j}(S)$, then

$$\psi = \sum_{j=1}^{k} \gamma_j^* \psi_j \gamma_j \in S_n(S).$$

Can define C*-convex hull.
There is a notion of **C*-extreme point** of a C*-convex set.

**Farenick (2000)** shows that the C*-extreme points of \((S_n(S))_{n \geq 1}\) coincide with the pure matrix states.
There is a notion of $C^*$-extreme point of a $C^*$-convex set.

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**Theorem (Farenick 2004)**

The C*-convex hull of the pure matrix states is BW-dense in the set of all matrix states.
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**Theorem (Farenick 2004)**

The C*-convex hull of the pure matrix states is BW-dense in the set of all matrix states.

Hence the pure matrix states completely norm S.
Putting it all together, we obtain:

**Theorem 3**

*Every operator system and every unital operator algebra has sufficiently many boundary representations.*
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*Every operator system and every unital operator algebra has sufficiently many boundary representations.*

**Corollary**

The $C^*$-envelope of every operator system and every unital operator algebra is obtained from a direct sum of boundary representations.
Where does this get us?

- Over four decades, we developed many techniques to get our hands on the C*-envelope of an operator algebra without using boundary representations.

I know of only a few examples where sufficiently many boundary representations are exhibited (Arveson, Muhly-Solel, D.-Katsoulis). The Choquet boundary, peak points and representing measures play a central role in the study of function algebras. Perhaps now, we can more diligently pursue the use of boundary representations in non-commutative dilation theory. This was central to Arveson's vision of the subject.
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The end.