Complete positivity of the map from a basis to its dual basis

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Joint work with Vern Paulsen, University of Houston
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Given a basis $\mathcal{B}$ of $M_n$, the *duality map* $D_\mathcal{B} : M_n \to M_n^d$ is the linear map that takes the basis $\mathcal{B}$ to its dual basis.
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**Theorem**

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Main question: for which bases $B$ is the duality map a complete order isomorphism?
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4. A source of “entanglement witnesses”: matrices that provide a test of entanglement.
Outline

For which bases $\mathcal{B}$ of $M_n$ is the duality map a complete order isomorphism?

Examples

Alternate characterizations of completely positive maps from $M_n$ to $M_p$

Expressions for the Choi matrix in non-standard bases
Definition

(Order on $M_p(M_n^d)$) A matrix of functionals $(f_{i,j}) \in M_p(M_n^d)$ belongs to $M_p(M_n^d)^+$ if and only if the evaluation map $\nu \mapsto (f_{i,j}(\nu))$ from $M_n$ to $M_p$ is completely positive map.
Definition

\[ \Phi : M_n \rightarrow M_n^d \] is a complete order isomorphism if \( \Phi \) is invertible and \( \Phi \) and \( \Phi^{-1} \) are completely positive.
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Definition

A linear map $\Psi : M_n \rightarrow M_n$ or $\Phi : M_n^d \rightarrow M_n$ is called a \textit{co-positive order isomorphism} provided that its composition $t \circ \Psi$ with the transpose map $t$ on $M_n$ is a complete order isomorphism.
Examples of complete or co-positive order isomorphisms

Definition
If $f \in M^d_n$, there is a unique matrix $D$ such that $f(X) = \text{tr}(DX)$ for all $X \in M_n$, and we call this matrix the density matrix for $f$, with no requirement of positivity for $f$ or $D$. 

Example
The map that takes a functional on $M_n$ to its density matrix is a co-positive order isomorphism.

Example
The map that takes a functional on $M_n$ to the conjugate transpose of its density matrix is a complete order isomorphism but is conjugate linear.
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The map that takes a functional on $M_n$ to the conjugate transpose of its density matrix is a complete order isomorphism but is conjugate linear.
$L(M_n)$ denotes the set of linear maps from $M_n$ to $M_n$.

**Definition**

Let $\mathcal{B}$ be a basis of $M_n$ and $\mathcal{E}$ the standard basis of matrix units, with a fixed order. A *change of basis map* is any linear map $C_{\mathcal{B}}$ in $L(M_n)$ taking the set $\mathcal{E}$ to the set $\mathcal{B}$.

**Definition**

By slight abuse of notation, we write $C_{\mathcal{B}}^T$ for the unique linear map in $L(M_n)$ whose matrix in the standard basis $\mathcal{E}$ is the transpose of the matrix of $C_{\mathcal{B}}$. We define $M_{\mathcal{B}} = C_{\mathcal{B}}C_{\mathcal{B}}^T \in L(M_n)$. 
Change of basis maps

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**Definition**

Let \(B\) be a basis of \(M_n\) and \(E\) the standard basis of matrix units, with a fixed order. A *change of basis map* is any linear map \(C_B\) in \(L(M_n)\) taking the set \(E\) to the set \(B\).

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By slight abuse of notation, we write \(C_B^T\) for the unique linear map in \(L(M_n)\) whose matrix in the standard basis \(E\) is the transpose of the matrix of \(C_B\). We define \(M_B = C_B C_B^T \in L(M_n)\).

The map \(M_B\) depends on the basis \(B\), but not on the particular choice of change of basis map \(C_B\).
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Main Theorem

Notation

If $C \in M_n$, then $\Phi_C : M_n \to M_n$ is the completely positive map defined by $\Phi_C(X) = CXC^*$. 

Theorem

Let $B$ be a basis of $M_n$. Then $\mathcal{D}_B$ is an order isomorphism iff $\mathcal{D}_B$ is either a complete order isomorphism or a co-positive order isomorphism. The former occurs iff there exists $C \in M_n$ such that $M_B = \Phi_C$, and the latter occurs iff $M_B = t \circ \Phi_C$ for some $C \in M_n$. 

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Examples: some bases closely related to the standard basis

Theorem

Let \((\lambda_{ij}) \in M_n\), with all \(\lambda_{ij}\) nonzero, and let \(B\) be the basis \(\{\lambda_{ij}E_{ij}\}\). Then \(D_B\) is an order isomorphism if and only if the matrix \((\lambda_{ij}^2)\) is positive semi-definite with rank one. In that case, \(D_B\) is a complete order isomorphism.
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Theorem

Let $(\lambda_{ij}) \in M_n$, with all $\lambda_{ij}$ nonzero, and let $\mathcal{B}$ be the basis $\{\lambda_{ij} E_{ij}\}$. Then $\mathcal{D}_\mathcal{B}$ is an order isomorphism if and only if the matrix $(\lambda_{ij}^2)$ is positive semi-definite with rank one. In that case, $\mathcal{D}_\mathcal{B}$ is a complete order isomorphism.

Example

If $C \in M_n$ is invertible, for the basis $\mathcal{B} = \{CE_{ij}C^*\}$, the duality map $\mathcal{D}_\mathcal{B}$ is a complete order isomorphism. In particular, if $\mathcal{B}$ is a system of matrix units then $\mathcal{D}_\mathcal{B}$ is a complete order isomorphism.
Definition

Let $e_0, \ldots, e_{n-1}$ be the standard basis of $\mathbb{C}^n$, and $\mathcal{B} = \{E_{ab} \mid a, b \in \mathbb{Z}_n\}$ the corresponding matrix units. Let $U, V \in M_n$ be defined by $Ve_j = z^j e_j$ and $Ue_j = e_{j+1}$ where $z = \exp(2\pi i / n)$ and $j \in \mathbb{Z}_n$. Then $\{\frac{1}{\sqrt{n}} U^a V^b \mid a, b \in \mathbb{Z}_n\}$ is an orthonormal basis for $M_n$ which we call the Weyl basis $\mathcal{W}$.

The unitary matrices $\{U^a V^b \mid a, b \in \mathbb{Z}_n\}$ are usually called the discrete Weyl matrices or the generalized Pauli matrices.
Theorem

For the Weyl basis $\mathcal{W}$, the duality map $D_{\mathcal{W}}$ is a complete order isomorphism if $n = 2$, and is not an order isomorphism for $n > 2$. 
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**Corollary**

For the basis of $M_{2^n}$ consisting of tensor products of the $2 \times 2$ Weyl basis, the duality map $D_\mathcal{W}$ is a complete order isomorphism.
Testing complete positivity using non-standard bases

Corollary

Let \( B = \{ B_j : 1 \leq j \leq n^2 \} \) be a basis for \( M_n \), and let \( \Psi : M_n \to M_p \) be a linear map.

1. If the duality map \( D_B \) is a complete order isomorphism, then \( \Psi \) is completely positive if and only if
   \[
   \sum_{j=1}^{n^2} \Psi(B_j) \otimes B_j \in (M_p \otimes M_n)^+.
   \]

2. If the duality map \( D_B \) is a co-positive order isomorphism, then \( \Psi \) is completely positive if and only if
   \[
   \sum_{j=1}^{n^2} \Psi(B_j) \otimes B_j^t \in (M_p \otimes M_n)^+.
   \]
A basis-free description of the Choi matrix

In the definition of $C_{\Phi}$,

$$C_{\Phi} = \sum_{ij} E_{ij} \otimes \Phi(E_{ij})$$

the basis $\{E_{ij}\}$ can’t be replaced by an arbitrary orthonormal basis. The following result provides an alternate description of the Choi matrix that does have this independence property. Given a matrix $B = (b_{i,j})$ we set $\overline{B} = (\overline{b_{i,j}})$. 
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Theorem

Let $\{B_l\}_{l=1}^{n^2}$ be an orthonormal basis for $M_n$, and $\Phi : M_n \to M_p$ linear. Then Choi’s matrix is given by

$$C_{\Phi} = \sum_{l=1}^{n^2} \overline{B_l} \otimes \Phi(B_l) \quad (1)$$
This expression shows that the Choi matrix is the partial transpose of a matrix defined by Jamiołkowski

$$\mathcal{J}(\Phi) = \sum_{ij} E_{ij}^* \otimes \Phi(E_{ij}).$$  \hspace{1cm} (2)

Jamiołski defined this correspondence as a tool in studying linear maps from $M_n$ into $M_p$. Here $E_{ij}$ could be replaced by any orthonormal basis, but positivity of $\mathcal{J}(\Phi)$ is not equivalent to complete positivity of $\Phi$. 
Vern Paulsen and Fred Shultz, Complete positivity of the map from a basis to its dual basis, arXiv:1212.4787