Quantum Expanders and Geometry of Operator Spaces

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A ⊂ B(H) C*-algebra
Consider E ⊂ A with dim(E) < ∞
Fix N ≥ 1
Then (Roger Smith): \( \forall a = [a_{ij}] \in M_N(E) \)
\[
\| [a_{ij}] \|_{M_N(E)} = \sup_{u: E \to M_N, \|u\|_{cb} \leq 1} \| [u(a_{ij})] \|_{M_N(M_N)}
\]

Natural question: How many u’s are needed?
**Natural question:** How many $u$’s are needed? 

Fix $\delta > 0$. We seek to minimize $\text{card}(\mathcal{T})$ over all **finite** sets 

$$\mathcal{T} \subset \{ u : E \to M_N | \|u\|_{cb} \leq 1 \}$$

such that 

$$\forall a = [a_{ij}] \in M_N(E) \quad (1 - \delta)\| [a_{ij}] \|_{M_N(E)} \leq \sup_{u \in \mathcal{T}} \| [u(a_{ij})] \|_{M_N(M_N)}$$

Or with $C = (1 - \delta)^{-1}$

$$\forall a = [a_{ij}] \in M_N(E) \quad \| [a_{ij}] \|_{M_N(E)} \leq C \sup_{u \in \mathcal{T}} \| [u(a_{ij})] \|_{M_N(M_N)}$$

Smallest $k$ denoted by $k_E(N, C)$
If $A$ is exact then: $\forall \delta > 0$, for all $E$ and all $N$ large enough, a single $u$ suffices:

$\exists u : E \rightarrow M_N$ with $\|u\|_{\text{cb}} \leq 1$ such that

$$\forall a = [a_{ij}] \in M_N(E), (1 - \delta)\|a_{ij}\|_{M_N(E)} \leq \|u(a_{ij})\|_{M_N(M_N)}$$
Using *metric entropy* in dimension $nN^2$ (note $\dim(CB(E, M_N)) = nN^2$), we find:

**Proposition**

\[ \exists \mathcal{T} \text{ with } \text{card}(\mathcal{T}) \leq \exp c_\delta nN^2 \]

such that

\[ \forall a = [a_{ij}] \in M_N(E) \quad (1 - \delta)\|a_{ij}\|_{M_N(E)} \leq \sup_{u \in \mathcal{T}} \|u(a_{ij})\|_{M_N(M_N)} \]

\[ \text{i.e. } k_E(N, C) \leq \exp c_\delta nN^2 \]
**Proof:** Indeed, there is a $\delta$-net $T$ in the unit ball of $CB(E, M_N)$ with

$$\text{card}(T) \leq (C/\delta)^{nN^2} \leq \exp c\delta nN^2.$$  

Thus $\forall u$ with $\|u\|_cb \leq 1$ $\exists v \in T$ such that $\|u - v\|_cb \leq \delta$. So:

$$\left\| [u(a_{ij})] \right\|_{M_N(M_N)} \leq \left\| [v(a_{ij})] \right\|_{M_N(M_N)} + \delta \left\| a_{ij} \right\|_{M_N(E)}$$

$$\leq \sup_{v \in T} \left\| [v(a_{ij})] \right\|_{M_N(M_N)} + \delta \left\| a_{ij} \right\|_{M_N(E)}$$

$$\left\| a_{ij} \right\| = \sup_{\|u\|_cb \leq 1} \left\| [u(a_{ij})] \right\|_{M_N(M_N)} \leq \sup_{v \in T} \left\| [v(a_{ij})] \right\|_{M_N(M_N)} + \delta \left\| a_{ij} \right\|$$

and hence

$$(1 - \delta)\left\| a_{ij} \right\|_{M_N(E)} \leq \sup_{v \in T} \left\| [v(a_{ij})] \right\|_{M_N(M_N)}$$
Thus:

$$\exists T \text{ with } \text{card}(T) \leq \exp c_\delta nN^2$$

**Goal:** A class of examples of $E$ for which this is essentially best possible, i.e.

**Theorem**

*For these spaces $E$, for any such $T$ for all $N$ large enough*

$$\text{card}(T) \geq \exp c'_\delta nN^2$$

Such spaces are “extremely not exact”
Including

$$E = \text{span}[U_1, \cdots, U_n] \subset C^*(\mathbb{F}_n)$$

and

$$E = OH_n$$

These are Operator space analogues of $\ell_1^n$ and $\ell_2^n$
The term “Quantum Expander” was introduced by Hastings and by Ben-Aroya and Ta-Shma in 2007 to designate a sequence \{U^{(N)} \mid N \geq 1\} of \(n\)-tuples \(U^{(N)} = (U_1^{(N)}, \ldots, U_n^{(N)})\) of \(N \times N\) unitary matrices such that there is an \(\varepsilon > 0\) satisfying the following “spectral gap” condition: \(\forall N \forall x \in M_N\) with \(\text{tr}(x) = 0\)

\[
\left\| \sum_{j=1}^{n} U_j^{(N)} x U_j^{(N)*} \right\|_2 \leq n(1 - \varepsilon) \|x\|_2,
\]

where \(\| \cdot \|_2\) denotes the Hilbert-Schmidt norm on \(M_N\).

Suffices to have this for infinitely many \(N\)’s i.e. \(N \in \{N(1) < N(2) < N(3) < \cdots\}\)

∃ Computer science motivation

An \(n\)-tuple \(U^{(N)}\) satisfying (1) will be called an \(\varepsilon\)-quantum expander.
The $\varepsilon$-quantum expander condition can be rewritten

$$\left\| \left( \sum_{j=1}^{n} U_j^{(N)} \otimes \overline{U_j^{(N)}} \right) (1 - P) \right\| \leq n(1 - \varepsilon)$$

where $P$ is the orthogonal projection onto $\mathcal{C}I$ ($I = \sum e_j \otimes \overline{e_j}$).
In analogy with the classical expanders, one seeks to exhibit (and hopefully to construct explicitly) sequences \( \{ U^{(N_m)} \mid m \geq 1 \} \) of \( n \)-tuples of \( N_m \times N_m \) unitary matrices that are \( \varepsilon \)-quantum expanders

\[
\left\| \left( \sum_{j=1}^{n} U_j^{(N_m)} \otimes \overline{U_j^{(N_m)}} \right) (1 - P) \right\| \leq n(1 - \varepsilon)
\]

with \( N_m \to \infty \) while \( n \) and \( \varepsilon > 0 \) remain fixed.
Quantum expanders = non-commutative version of classical expanders

When \( G \) is a finite group generated by \( S = \{ t_1, \cdots, t_n \} \) the associated Cayley graph \( G(G, S) \) is said to have a spectral gap if the left regular representation \( \lambda_G \) satisfies

\[
\left\| \sum \lambda_G(t_j) \right\|_\perp < n(1 - \varepsilon) \tag{2}
\]

where \( \perp \) denotes the constant function 1 on \( G \).

Obviously, this is equivalent to the condition that the unitaries \( U_j = \lambda_G(t_j) \) satisfy (here \( N = |G| \))

\[
\left\| \sum_1^n U_j^{(N)} x U_j^{(N)*} \right\|_2 \leq n(1 - \varepsilon) \|x\|_2
\]

when restricted to diagonal matrices \( x \) with \( \text{tr}(x) = 0 \)
A sequence of Cayley graphs $G(G^{(m)}, S^{(m)})$ constitutes an expander in the usual sense if

$$\| \sum_{j=1}^{n} \lambda_{G(t_j)} \|_{\| \|} < n(1 - \varepsilon)$$

is satisfied with $\varepsilon > 0$ and $n$ fixed while $|G^{(m)}| \to \infty$.

Expanders (equivalently expanding graphs) have been extremely useful, especially (in the applied direction) since Margulis 1973 and Lubotzky-Phillips-Sarnak 1988 (“Ramanujan graphs”) obtained explicit constructions. But there are also useful random ones, cf. Joel Friedman’s work Memoirs AMS 2008.

Moreover:

\( G \) a finite group generated by \( S = \{t_1, \cdots, t_n\} \)

Assume

\[
\| \sum \lambda_G(t_j) \|_{\mathbb{I}^\perp} < n(1 - \varepsilon)
\]

Then for any non trivial \textbf{irreducible} representation \( \pi \) of \( G \)

\[
U_j = \pi(t_j)
\]

is an \( \varepsilon \)-quantum expander i.e.

\[
\| \sum U_j \otimes \overline{U}_j (1 - P) \| \leq n(1 - \varepsilon)
\]

\textbf{Proof:} \( [\pi \otimes \overline{\pi}]_{\mathbb{I}^\perp} \subset \lambda_G |_{\mathbb{I}^\perp} \)
Moreover, if $\sigma \neq \pi$ is another irreducible representation, let

$$U_j = \pi(t_j) \quad V_j = \sigma(t_j)$$

Then

$$\| \sum U_j \otimes V_j \| \leq n(1 - \varepsilon)$$

(“$\varepsilon$-separated”)

**Proof:** $[\pi \otimes \bar{\sigma}] \subset \lambda_{G|\mathbb{I}^\perp}$

One can show (de la Harpe-Robertson-Valette 1993) that in the presence of an $n$-element $\varepsilon$-expander in $G$ ($n, \varepsilon > 0$ fixed)

$$\forall N' \quad \text{card}\{\pi \mid d_{\pi} \leq N'\} \leq \exp (c_\varepsilon nN'^2)$$

**Open problem (Meshulam-Wigderson):**
Is this bound optimal?
More generally for any subset

\[ \mathcal{T} \subset U(N)^n \]

such that

\[ \forall u \neq v \in \mathcal{T} \quad \left\| \sum_j u_j \otimes \bar{v}_j \right\| \leq n(1 - \varepsilon) \]

("\varepsilon\)-separated")

we must have

\[ \text{card}(\mathcal{T}) \leq \exp(c_\varepsilon nN^2) \]

\(c_\varepsilon\) depending only on \(\varepsilon > 0\)

**Proof:** easy metric entropy argument: ambient dimension is

\[ nN^2 \]
Main result

Theorem

\[ \exists \beta > 0 \; \exists \delta > 0 \text{ such that } \]

for each \(0 < \varepsilon < 1\), for all sufficiently large integers \(n\) and \(N\),

(precisely \(\forall n \geq n_0(\varepsilon)\) and \(N \geq N_0(\varepsilon, n)\))

there is a \(\delta\)-separated family \(\{u(t) \mid t \in T\} \subset U(N)^n\) formed of \(\varepsilon\)-quantum expanders such that

\[ |T| \geq \exp \beta nN^2. \]
We make crucial use of a result due to Hastings 2007:

**Lemma (Hastings)**

If we equip $U(N)^n$ with its normalized Haar measure $\mathbb{P}$, then for each $n$ and $\varepsilon > 0$

$$\lim_{N \to \infty} \mathbb{P}\{u = (u_j) \in U(N)^n \mid \|\sum_j u_j \otimes \overline{u_j}\|_\perp \leq 2\sqrt{n - 1} + \varepsilon n\} = 1.$$

**Best possible:** This Lemma fails if $2\sqrt{n - 1}$ is replaced by any smaller number. However, our paper includes a quicker proof of a result that suffices for our needs (where $2\sqrt{n - 1}$ is replaced by $4C\sqrt{n}$, $C$ being a numerical constant).
Definition

Fix $\delta > 0$. We will say that $x, y$ in $M_N^n$ are $\delta$-separated if

$$\| \sum x_j \otimes \bar{y}_j \| \leq (1 - \delta) \| \sum x_j \otimes \bar{x}_j \|^{1/2} \| \sum y_j \otimes \bar{y}_j \|^{1/2}.$$ 

A family of elements is called $\delta$-separated if any two distinct members in it are $\delta$-separated.

Recall

$$\| \sum x_j \otimes \bar{y}_j \| = \sup \{ \| \sum x_j \xi y_j^* \|_2 \mid \| \xi \|_2 \leq 1 \}$$
\forall x, y \in M^n_N

\begin{align*}
x.y &= \sum x_j \otimes y_j \quad x.\bar{y} = \sum x_j \otimes \bar{y}_j
\end{align*}

With this notation
\begin{align*}
\delta \text{-separated means}
\|x.\bar{y}\| \leq (1 - \delta)\|x.\bar{x}\|^{1/2}\|y.\bar{y}\|^{1/2}
\end{align*}

Note that if \( x \in U(N)^n \) i.e. \( x_j \) is unitary then
\[
\|x.\bar{x}\| = n
\]

because \( \sum_{j=1}^n x_j I x_j^* = nI \)
### Operator spaces

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&quot;Non-commutative Banach spaces&quot;) An operator space is a subspace of $B(\mathcal{H})$, i.e. we are given $E \subset B(H)$</td>
</tr>
</tbody>
</table>

Morphisms: CB maps

**Fundamental result:** Arveson version of Hahn-Banach CB maps, Acta 1969

“Operator space Theory" is now well developed after Ruan’s 1987 thesis cf. Effros-Ruan, Blecher-Paulsen, and many more...

cf. one book by Effros-Ruan, & one by myself
Let

\[ u : E \to F \]

be a linear map between operator spaces. We denote for any given \( N \geq 1 \)

\[ u_N = \text{Id} \otimes u : M_N(E) \to M_N(F) \]

\[ [a_{ij}] \mapsto [u(a_{ij})] \]

\[ (u_N = \text{Id} \otimes u : M_N \otimes E \to M_N \otimes F) \]

Recall that

\[ \|u\|_{cb} = \sup_{N \geq 1} \|u_N\|. \]
For an operator space the norm is replaced by a sequence of norms

\[ \{ \| \cdot \|_{M_N(E)} \mid N \geq 1 \} \]

The ordinary norm on \( E \) corresponds to \( N = 1 \)
Given

\[ E \subset B(H) \quad F \subset B(\mathcal{H}) \]

we define

\[ E \otimes_{\text{min}} F \subset B(H \otimes_2 \mathcal{H}) \]

(“spatial” or “minimal” tensor product)

Note: This norm will be used everywhere!
A first application related to non-compactness/non-separability of the metric space of $n$-dim operator spaces (Junge-P, 1994)

**Theorem**

There are numbers $\beta_1 > 0$, $\delta_3 > 0$, $n_0 > 1$ and a function $n \mapsto N_0(n)$ from $\mathbb{N}$ to itself such that for any $n \geq n_0$ and $N \geq N_0(n)$, there is a family $\{E_t \mid t \in T_1\}$ of $n$-dimensional subspaces of $M_N$, with cardinality $|T_1| \geq \exp \beta_1 nN^2$, such that for any $s \neq t \in T_1$ we have

$$d_{cb}(E_s, E_t) > 1 + \delta_3.$$
An application with Geometric flavor

Operator space analogue of Hilbert space (or Euclidean space)

\[ \mathcal{O}H = \overline{\text{span}}[\theta_j \mid j \in \mathbb{N}] \subset B(H) \]

is analogous to \( \ell_2 \)

\[ \mathcal{O}H \text{ characterized by:} \]

For any \( N \) and any (finite) sequence \( x = (x_j) \) with \( x_j \in M_N \)

\[ \left\| \sum x_j \otimes \theta_j \right\| = \left\| \tilde{x} \cdot x \right\|^{1/2} = \sup_{y : \|\tilde{y} \cdot y\| \leq 1} \|\tilde{y} \cdot x\|. \]

Similarly:

\[ \mathcal{O}H_n = \overline{\text{span}}[\theta_j \mid 1 \leq j \leq n] \subset B(H) \]

is analogous to \( \ell_2^n \)
Definition of $OH$ uses Haagerup’s Cauchy-Schwarz inequality

$$\forall x = (x_j) \in B(H)^{(\mathbb{N})} \quad \|\bar{y}.x\| \leq \|\bar{y}.y\|^{1/2} \|\bar{x}.x\|^{1/2}$$

and hence $\exists \theta_j$ such that

$$\| \sum x_j \otimes \theta_j \| = \|\bar{x}.x\|^{1/2} = \sup_{y: \|\bar{y}.y\| \leq 1} \|\bar{y}.x\|.$$ 

In particular, if $\dim(H) = N$ we have

$$\forall x = (x_j) \in M_N^n \quad \| \sum_{1}^{n} x_j \otimes \theta_j \| = \|\bar{x}.x\|^{1/2} = \sup_{y: \|\bar{y}.y\| \leq 1} \|\bar{y}.x\|.$$ 

When $N = 1$, we recover the classical formula

$$\| \sum x_j \otimes e_j \| = \langle x, x \rangle^{1/2} = \sup_{y: \langle y, y \rangle \leq 1} |\langle y, x \rangle|.$$
Classical Geometric problem
Consider a symmetric convex body $B = B_E$
Unit ball of an $n$-dimensional normed space $E$
Given a constant $C > 1$, estimate the minimal number
$k = k_E(C)$ of functionals $f_1, \cdots f_k$ in the dual $E^*$ such that

$$\forall x \in E \quad \sup_{1 \leq j \leq k} |f_j(x)| \leq \|x\|_E \leq C \sup_{1 \leq j \leq k} |f_j(x)|.$$
Geometrically (in the real case): $B_{E^*} \overset{C}{\sim} \text{conv}\{\pm f_j\}$

$P = \text{conv}\{\pm f_j\} = \text{polyhedron with at most } 2k \text{ vertices}$

So the polar, $P^*$, has at most $2k$ faces.

**Example:** the $n$-dimensional cube has $2^n$ vertices and $2n$ faces

When $E$ has (real) dimension $n$ it is well known that

$$\forall C = 1 + \delta \in [1, 2] \quad k_E(C) \leq \exp\left(\frac{4}{\delta}n\right).$$
Extreme cases:

- Maximal order of growth:
  \[ E = \ell_2^n \text{ (or } \ell_p^n \text{ for } 1 \leq p < \infty \text{ or uniformly convex)} \]
  \[ k_E(1 + \delta) \geq \exp(c_\delta n). \]

- Minimal order of growth:
  \[ k_E(C) = n \text{ for } E = \ell_\infty^n \]
Let $E$ finite dim. Banach space. Fix a constant $C > 1$, Recall 

$$k_E(C)$$

is the minimal number $k$ of functionals $f_1, \cdots f_k \in B_{E^*}$ such that 

$$\|x\|_E \leq C \sup_{1 \leq j \leq k} |f_j(x)|.$$

Equivalently:

$$k_E(C) = \inf \{ k \mid E_C \subseteq \ell_k^\infty \}$$
Matricial version of $k_E(C)$

Let $E$ be a finite dimensional operator space. Fix $C > 0$. We denote by

$$k_E(N, C)$$

the smallest $k$ such that there are linear maps

$$f_j : E \to M_N \quad (1 \leq j \leq k)$$

with $\|f_j\|_{cb} \leq 1$ satisfying $\forall x \in M_N(E)$

$$\|x\|_{M_N(E)} \leq C \sup_{1 \leq j \leq k} \|(\text{Id} \otimes f_j)(x)\|_{M_N(M_N)}.$$ 

Equivalently:

$$k_E(N, C) = \inf\{k \mid E \subset \ell_\infty^k \otimes M_N\}.$$
Theorem

There are numbers $C_1 > 1$, $b > 0$ such that for any $n$, $N$ large enough we have

$$k_{OH_n}(N, C_1) \geq \exp bnN^2.$$  

Compare: Universal UPPER bound:

for any $n$-dimensional $E$

for all $N$ large enough

$$k_E(N, C_1) \leq \exp b'nN^2$$

(because $\dim(M_N(E)) = nN^2$)
We will recall a classical argument for the Euclidean case, i.e. the case $N = 1$. Let $E = \ell_2^n$. Let $C = (1 - \delta)^{-1}$ for some $0 < \delta < 1$. We will show

$$k_E(C) \geq \exp cn$$

Two ingredients:

(i) **Uniform smoothness** of $E^*$: For any given point $t \in B_E$, and any $f \in B_E$

$$\langle f, t \rangle \geq 1 - \delta \implies \|f - t\| \leq \sqrt{2\delta}$$

(ii) **Large metric entropy** (valid for any $E$): There is a subset $T \subset S_E$ with $|T| \geq \exp c\epsilon' n$ such that

$$\forall s \neq t \in T \times T \quad \|s - t\| \geq \epsilon'$$

$$\Leftrightarrow \Re(\bar{s}.t) \leq 1 - \epsilon'^2 / 2$$
Proof that $k = k_E(C) \geq \exp cn$: Recall $C = (1 - \delta)^{-1}$.

$$\forall x \in E \sup_{1 \leq j \leq k} |f_j(x)| \leq \|x\|_E \leq (1 - \delta)^{-1} \sup_{1 \leq j \leq k} |f_j(x)|.$$ 

Real case for simplicity: changing $k$ to $2k$ (consider $\pm f_j$):

$$\forall x \in E \sup_{1 \leq j \leq 2k} f_j(x) \leq \|x\|_E \leq (1 - \delta)^{-1} \sup_{1 \leq j \leq 2k} f_j(x).$$

Take $x = t$. For any $t \in T$, there is $j(t)$ so that

$$1 - \delta \leq f_{j(t)}(t)$$

But by the smoothness

$$(*) \quad \|f_{j(t)} - t\| \leq \sqrt{2\delta}$$

Now choosing $\delta < \varepsilon'^2/32$ and recalling $\|s - t\| \geq \varepsilon'$ for $s \neq t$

$$(*) \Rightarrow \|f_{j(t)} - f_{j(s)}\| \geq \varepsilon' - 2\sqrt{2\delta} > \varepsilon'/2 > 0$$

and hence $j(s) \neq j(t)$ whenever $s \neq t$.

Conclusion: $2k \geq |T|$ and hence $2k \geq \exp c_{\varepsilon'}n$. Q.E.D.
We follow this proof to show

**Theorem**

*There are numbers $C_1 > 1$, $b > 0$ such that for any $n, N$ large enough we have*

$$k_{OH_n}(N, C_1) \geq \exp bnN^2.$$  

Proof requires *matricial* analogues of (i) and (ii)
Recall $\|\bar{y}.x\| = \|\sum \bar{y}_j \otimes x_j\|

We will need the analogue of smooth points on the sphere

Consider $x = (x_j) \in M_N^n$ in the “unit sphere” i.e. such that $\|\bar{x}.x\| = 1$.

Let $Orb(y) = \{y' = (y'_j) \mid \exists u, v \in U(N) \ y'_j = uy_jv\}$

Recall $1 = \sup_{y: \|\bar{y}.y\| \leq 1} \|\bar{y}.x\|$. 

Note $\|\bar{y}.x\| = 1 \implies \|\bar{y}'.x\| = 1 \forall y' \in Orb(y)$

We say that $x = (x_j) \in M_N^n$ with $\|\bar{x}.x\| = 1$ is $M_N$-smooth if

$$\|\bar{y}.y\| \leq 1 \quad \|\bar{y}.x\| = 1 \implies y \in Orb(x)$$

**Lemma (Matricial uniform smoothness)**

Consider $x = (x_j) = n^{-1/2}(u_j)$ with $(u_j) \in U(N)^n$. Observe $\|\bar{x}.x\| = 1$. Then, $x$ is $M_N$-smooth IFF $(u_j)$ is an $\varepsilon$-quantum expander for some $\varepsilon > 0$. Moreover, a lower bound on $\varepsilon$ ensures “uniform smoothness” as in (ii).
Thus we need to work with points in the sphere of $M_N(OH_n)$ of the form: $t = \sum_j t_j \otimes \theta_j$ with $\| \tilde{t}.t \| = 1$ that in addition form $\varepsilon$-quantum expanders

We identify such a point with $s = (s_j) \in M^N_N$

We will use $t_u = (t_j) = n^{-1/2}(u_j)$ with $(u_j) \in U(N)^n$

$t_u$ is just $u$ after renormalization

**Lemma (Matricial metric entropy)**

There are numbers $\varepsilon' > 0$, $b > 0$ such that for any $0 < \varepsilon < 1$ for any $n, N$ large enough there is a subset

$$T \subset U(N)^n \quad \text{with} \quad |T| \geq \exp bnN^2$$

formed of $\varepsilon$-quantum expanders that are $\varepsilon'$-separated i.e.

$$\forall u \neq v \in T \quad n^{-1}\|\tilde{u}.v\| = \|\tilde{t}_u.t_v\| \leq 1 - \varepsilon'^2/2$$

and a fortiori

$$\left( n^{-1}\sum_{j=1}^n \|u_j - v_j\|_{L^2(\tau_N)}^2 \right)^{1/2} \geq \varepsilon'$$
With these lemmas, we can run the proof of

$$k_{OH_n}(N, C_1) \geq \exp bnN^2$$

exactly as we did for the classical case $N = 1$
Similarly, in addition to \( E = OH_n \), we have

\[
\exp b'nN^2 \geq k_E(N, C_1) \geq \exp bnN^2
\]

for

\[
E = \ell_1^n \subset C^*(\mathbb{F}_n)
\]

and for

\[
E = R_n + C_n
\]
Connection with Exactness

This leads to a natural class: the operator spaces (or $C^*$-algebras) $X$ such that for some $C$ for any $E \subset X$ we have

$$\limsup_{N \to \infty} \frac{\log k_E(N, C)}{N^2} = 0$$

We call these (matricially) subGaussian.

They should verify a sort of “generalized exactness”

Unfortunately we could obtain significant results only with a stronger property

$$\limsup_{N \to \infty} \frac{\log k_E(N, C)}{N} = 0$$

called (matricially) subexponential.
Indeed in “(matricially) subexponential” spaces
Behaviour /Gaussian random matrices similar to exact case
⇒ **Subexponential Operator spaces satisfy the OSGT**
Extends the Grothendieck’s Theorem from Shlyakhtenko-P 2002 from exact to subexponential
A variant: $K_E(C, N)$

$K_E(N, C)$ the smallest $K$ such that there is a single map

$$u : E \to M_K \quad \text{with} \quad \|u\|_{cb} \leq 1$$

satisfying

$$\forall a = [a_{ij}] \in M_N(E) \quad \|[a_{ij}]\|_{M_N(E)} \leq C\|[u(a_{ij})]\|_{M_N(M_K)}$$

Equivalently (instead of $E \subset \ell_k^\infty \otimes M_N$) we have

$$E \subset M_K$$

Note obviously

$$K_E(N, C) \leq Nk_E(N, C)$$

So $k_E(N, C)$ subexponential implies $K_E(N, C)$ subexponential
A exact ⇔ ∃C ∀E ⊂ A \sup_N K_E(N, C) < \infty

But we have examples of C*-algebras A such that K_E(N, 1 + \varepsilon) has subexponential growth (even polynomial growth) for any \varepsilon > 0 and any finite dim. E ⊂ A but A is not exact
Let

\[ U_j(\omega) = \bigoplus_N U_j^{(N)} \subset \bigoplus_N M_N \]

Let

\[ A_\omega = C^* < U_j(\omega), 1 \leq j \leq 3 > \]

**Theorem**

*For almost all* \( \omega \), \( A_\omega \) *is subexponential*

*i.e. \( \forall C > 1 \forall E \subset A \lim sup N^{-1} \log K_E(N, C) = 0 *but not exact.*

Initially: Gaussian case (using Haagerup-Thorbjoernsen)
Later: Unitary case: de la Salle (using Collins-Male)
Thank you!