Isomorphisms of Lattices of Bures-Closed Bimodules over Cartan MASAs

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Cartan Pairs & Main Question

Throughout, \((\mathcal{M}, \mathcal{D})\) will be a Cartan pair, i.e. a \(W^*\)-algebra \(\mathcal{M}\) and a MASA \(\mathcal{D} \subseteq \mathcal{M}\) such that

- \(\overline{\text{span}}^{\text{wk}-}\{U \in \mathcal{U}(\mathcal{M}) : UDU^* = \mathcal{D}\} = \mathcal{M}\)
- \(\exists\) a faithful, normal, cond. expect’n \(E : \mathcal{M} \to \mathcal{D}\).

The \textbf{Bures topology} on \(\mathcal{M}\) is the topology generated by the seminorms \(\mathcal{M} \ni T \mapsto \sqrt{\omega(E(T^*T))}\), where \(\omega \in \mathcal{D}_{++}\).

\(\mathcal{B}(\mathcal{M}, \mathcal{D})\) denotes lattice of Bures-closed \(\mathcal{D}\)-bimodules

\textbf{Main Question}

\textit{For Cartan pairs} \((\mathcal{M}_i, \mathcal{D}_i)\) \((i = 1, 2)\), \textit{when are} \(\mathcal{B}(\mathcal{M}_1, \mathcal{D}_1)\) \textit{and} \(\mathcal{B}(\mathcal{M}_2, \mathcal{D}_2)\) \textit{isomorphic lattices?}
Motivation

When $\mathcal{M}_*$ is sep, Feldman-Moore showed $\exists$ measurable equiv. relation $R$ & a 2-cocycle $\sigma$ on $R$ s. t.

$$\mathcal{M} = \mathbf{M}(R, \sigma) \quad \& \quad \mathcal{D} = \mathbf{A}(R, \sigma).$$

Muhly-Saito-Solel (1988) asserted that the weak-$\ast$ closed $\mathcal{D}$-bimod’s can be described by “measure theoretic” data (certain measurable subsets of $R$).

Unfortunately: Validity of M-S-S assertion if $\mathcal{M}$ not hyperfinite is unknown.

When $\mathcal{M}$ hyperfinite, M-S-S is valid (Fulman, 1997).

However: Cameron-Pitts-Zarikian (2013) described Bures-closed $\mathcal{D}$-bimod’s. ($\mathcal{M}_*$ not nec. sep), which led to Main Question.
Fix a faithful, normal, semi-finite weight $\phi$ on $\mathcal{M}$ with $\phi = \phi \circ E$; & let

$(\mathcal{H}, \pi, \eta)$ be the semi-cyclic rep’n. for $\phi$.

Recall:

- $\mathcal{H} := L^2(\mathcal{M}, \phi)$ (completion of $\{x \in \mathcal{M} : \phi(x^*x) < \infty\}$);
- $\eta : \mathcal{M} \cap L^2(\mathcal{M}, \phi) \rightarrow \mathcal{H}$ is inclusion map.
- $\pi$ is left reg. rep’n of $\mathcal{M}$ on $\mathcal{H}$;

Define 2 Repn’s of $\mathcal{D}$ on $\mathcal{H}$:

- $\pi_\ell$ is left reg rep’n of $\mathcal{D}$ on $\mathcal{H}$ (i.e. $\pi_\ell = \pi|_\mathcal{D}$);
- $\pi_r$ is right reg rep’n of $\mathcal{D}$ on $\mathcal{H}$ (well-def’ned because $\phi = \phi \circ E$).

Also, let $P \in B(\mathcal{H})$ be the proj’n from $\eta(x) \mapsto \eta(E(x))$. 
For $\mathcal{M}_\ast$ sep., Feldman-Moore proved the following, but it holds in general too.

**Theorem (C-P-Z (maybe others))**

Let $\mathcal{Z} := (\pi_\ell(D) \cup \pi_r(D))''$. Then $\mathcal{Z}$ is a MASA in $\mathcal{B}(\mathcal{H})$. 
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Example of above concepts

Consider \((B(\ell^2(N)), \ell^\infty(N))\), and \(\phi\) the tracial weight. Then

- \(\mathcal{H}\) = Hilbert-Schmidt op’s;
- \(\pi(T)\) is left multiplication (of a H-S op) by \(T \in B(\ell^2(N))\),
- \(\pi_\ell\) and \(\pi_r\) are left and right multiplication of H-S op’s by diagonal matrices.
- \(\mathcal{Z}\) is all bounded “Schur multipliers” on the H-S op’s;
- \(P\) takes a H-S op. to its diagonal part.

Also, a net \(T_\lambda \overset{\text{Bures}}{\to} 0\) iff \(\|T_\lambda D\|_{\text{H.S.}} \to 0\) for all \(D \in \text{range}(P) = \text{diag}(\ell^2(N))\).
**Definitions of Supp & Bimod**

**FACT:** For any \((\mathcal{M}, \mathcal{D})\), if \(C \subseteq \mathcal{M}\) convex, then 
\[ C^{\text{wk-}*} \subseteq C^{\text{Bures}}. \]

**Definition**

- For \(S \subseteq \mathcal{M}\) a \(\mathcal{D}\)-bimod, \(\pi(S)\)range\((P) \in \text{Lat}(\mathcal{Z})\), define \(\text{supp}(S) \in \mathcal{Z}\) to be the proj’n onto this space.
- For \(T \in \mathcal{M}\), \(\text{supp}(T) := \text{supp}(\text{bimodule gen by } T)\).
- For \(Q \in \text{proj}(\mathcal{Z})\), let 
\[ \text{bimod}(Q) := \{ T \in \mathcal{M} : \text{supp}(T) \leq Q \}. \] Then \(\text{bimod}(Q)\) is a Bures (\& hence \(\text{wk-}*\)) closed bimodule in \(\mathcal{M}\).

When \(\mathcal{M}_*\) sep., and \(\mathcal{M}\) identified with its Feldman-Moore rep’n, \(\text{supp}(S)\) “is” the meas.-th’tic support of \(S\).
**Description of Bures-Closed Bimodules**

**Theorem (C-P-Z)**

\[ S \subseteq \mathcal{M} \text{ a } \mathcal{D} \text{-bimodule. TFAE:} \]

1. \( S = \text{bimod}(\text{supp}(S)) \);
2. \( S \) is Bures-closed;

**Consequence:**

**Corollary (C-P-Z)**

\( \mathcal{B}(\mathcal{M}, \mathcal{D}) \text{ and } \text{proj}(\mathcal{Z}) \) are isomorphic lattices.
Aside: Reformulation of M-S-S Assertion

For $S$ a $D$-bimodule, easy to see that

$$S^{wk-*} \subseteq \text{bimod}(\text{supp}(S)).$$

Assertion of Muhly-Saito-Solel becomes

When $\mathcal{M}^\ast$ is separable, then $\subseteq$ is equality, i.e.

$$S^{wk-*} = S^{\text{Bures}}.$$
Main Result

Notation

For any Cartan pair, \( \text{card}_a(\mathcal{M}, \mathcal{D}) \) denotes cardinality of \( R_a := \{(Q_1, Q_2) \in \text{atom}(\mathcal{D}) \times \text{atom}(\mathcal{D}) : Q_1 \mathcal{M} Q_2 \neq (0)\} \).

FACT: Elt’s of \( R_a \) correspond to min elt’s of \( \mathcal{B}(\mathcal{M}, \mathcal{D}) \).

Answer to Main Question (Fuller-P, 2013)

Let \((\mathcal{M}_i, \mathcal{D}_i)\) be Cartan pairs with \( \mathcal{M}_{i*} \) separable. TFAE

1. \( \mathcal{B}(\mathcal{M}_1, \mathcal{D}_1) \) and \( \mathcal{B}(\mathcal{M}_2, \mathcal{D}_2) \) are isomorphic lattices.
2. \( \text{proj}(\mathcal{Z}_1) \) and \( \text{proj}(\mathcal{Z}_2) \) are isomorphic lattices.
3. \( \mathcal{Z}_1 \) and \( \mathcal{Z}_2 \) are isomorphic \( W^* \)-algebras.
4. \( \text{card}_a(\mathcal{M}_1, \mathcal{D}_1) = \text{card}_a(\mathcal{M}_2, \mathcal{D}_2) \).
Examples

Very different pairs have isomorphic Bures lattices:

**Non-Atomic Example**

If $(\mathcal{M}_i, \mathcal{D}_i)$ are (sep. acting) Cartan pairs with $\mathcal{D}_i$ non-atomic, then $\mathcal{B}(\mathcal{M}_1, \mathcal{D}_1) \simeq \mathcal{B}(\mathcal{M}_2, \mathcal{D}_2)$ because $\mathcal{D}_i \simeq L^\infty([0, 1], m)$.

In particular, can take $\mathcal{M}_1$ type II and $\mathcal{M}_2$ type III.

**Simple Atomic Example**

Let $\mathcal{M}_1 = M_2(\mathbb{C})$, $\mathcal{D}_1 = D_2(\mathbb{C})$ and $\mathcal{M}_2 = \mathcal{D}_2 = \ell^\infty(4)$.

Then $(\mathcal{M}_1, \mathcal{D}_1)$ & $(\mathcal{M}_2, \mathcal{D}_2)$ Cartan pairs with $\mathcal{B}(\mathcal{M}_1, \mathcal{D}_1) \simeq \mathcal{B}(\mathcal{M}_2, \mathcal{D}_2)$ because $\mathcal{Z}_i \simeq \ell^\infty(4)$. 
THANK YOU!