Local and Global Aspects of Time-Frequency Analysis With Applications to Sound Analysis

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Outline

1. Time-frequency (TF) analysis as part of applied harmonic analysis - a conceptual introduction
   - Local aspects
   - Global aspects
2. Adaptive representations: introducing flexibility in TF representations
3. TF localization operators: spectral properties and discretization
4. Applications
5. Summary and Perspectives
Time-frequency (TF) analysis

Overarching idea: try to efficiently represent a function/signal $f$ with few and simple basic elements $g_j$ by

$$f = \sum_j c_j g_j.$$
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Audio: What can an appropriate signal representation provide?
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Time, Frequency – and Time-Frequency

\[ f(t) = \sum_{n} c_n \delta_n(t) = \sum_{k} \hat{c}_k e^{2\pi i k \omega_0 t} \]
Time, Frequency – and Time-Frequency

\[ f(t) = \sum_{n} c_n \delta_n(t) = \sum_{k} \hat{c}_k e^{2\pi i k \omega_0 t} = \sum_{n,k} \tilde{c}_{n,k} g_{n,k}(t) \]
Measuring time-frequency energy: STFT

Definition: short-time Fourier transform (STFT)

\[ \mathcal{V}_g f(z) = \mathcal{V}_g f(x, \xi) = \int_{\mathbb{R}} f(t)g(t-x)e^{-2\pi i \xi t} dt \]  \hspace{1cm} (1)

Short-time Fourier transform of \( f \) with respect to a window function \( g \in L^2(\mathbb{R}) \), \( z = (x, \xi) \in \mathbb{R}^2 \).
Measuring time-frequency energy: STFT

**Definition: short-time Fourier transform (STFT)**

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Short-time Fourier transform of \( f \) with respect to a window function \( g \in L^2(\mathbb{R}) \), \( z = (x, \xi) \in \mathbb{R}^2 \).

Time-frequency representations typically ambiguous: what we measure depends on how we measure. On the other hand, for all \( g \in L^2 \) have:

\[ f = \frac{1}{\|g\|^2_2} \int_{\mathbb{R}^2} \mathcal{V}_g f(z) \pi(z) g \, dz. \]

Here: \( \pi(z) g(t) = M_\xi T_x g(t) = g(t-x) e^{2\pi i \xi t} \)
Local aspects: uncertainty and concentration

Measure locally in time - ok, but what happens in frequency?
Local aspects: uncertainty and concentration

Measure locally in time - ok, but what happens in frequency?
Local aspects: uncertainty and concentration

Measure locally in time - ok, but what happens in frequency?

- Signal and localized signal (red)
- Positive frequency components
Localization: $f \rightarrow f \cdot m$, where $m$ is some compactly supported localizing function (often called window).

Then: $f \cdot m = \hat{f} \ast \hat{m}$. ($\ast$ denotes convolution)

In our example: $m = box$ (boxcar function) and $\hat{m}(\xi) = sinc(\xi) = \frac{\sin(\xi)}{\xi}$.

More generally (Heisenberg’s) uncertainty principle prohibits arbitrary precision simultaneously in time and frequency.
Gabor’s idea: use window with optimal simultaneous localization in time \textit{and} frequency - the Gaussian window \( g(t) = e^{-\pi x^2} \), and pave the time-frequency plane with its time-frequency shifts on \( \mathbb{Z} \times \mathbb{Z} \):
However, a system constructed according to Gabor’s idea, with a Gaussian function, cannot form an orthonormal (or even Riesz) basis; nor with any other ”nice window”...
Global aspects: energy conservation

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**Definition: Frames**

A sequence \( \{g_j, j \in J\} \) in \( L^2(\mathbb{R}) \) is a frame if for some positive constants \( A, B > 0 \) and all \( f \in L^2(\mathbb{R}) \)

\[
A \|f\|_2^2 \leq \sum_{j \in J} |\langle f, g_j \rangle|^2 \leq B \|f\|_2^2.
\]

– Good time-frequency resolution requires redundancy! –

Unconditional convergence of frame expansions.
Global aspects: energy conservation

Gabor frames (Weyl-Heisenberg frames, time-frequency frames) are given, according to Gabor’s idea, by time-frequency shifts of a window $g$ along a lattice $\Lambda$, which replaces $\mathbb{Z} \times \mathbb{Z}$ and is slightly more dense in general:

**Definition: Gabor Frames**

For a (non-zero) window function $g$ and parameters $a, b > 0$, the set of time-frequency shifts of $g$ along the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, given as

$$\mathcal{G}(g, \Lambda) = \{M_{bk} T_{an}g : k, n \in \mathbb{Z}\}$$

$$= \{\pi(\lambda)g : \lambda \in \Lambda\}$$

is called a *Gabor system*. If $\mathcal{G}(g, \Lambda)$ is a frame, it is called a *Gabor frame*. 
Local aspects: uncertainty and concentration

To obtain different time-frequency resolution, different window eccentricity (dilation) may be chosen:

Salient coefficients for long window

Salient coefficients for short window

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Local and Global Aspects of Time-Frequency Analysis
Local aspects: uncertainty and concentration

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Local aspects: uncertainty and concentration

Sometimes, different window shape desired in different parts of the time-frequency plane to provide good representations for various signal components:
Introducing Flexibility

In order to overcome some of the limitations imposed by the uncertainty principle, one can introduce more flexible construction principles:

1. Use various different windows
2. Use more flexible, e.g. adaptive sampling set
3. Combine both
Introducing Flexibility

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3. Combine both

Main mathematical problems involved:

1. Frame property and characterization of function spaces by frame coefficients - **global** characterization.
2. Invertibility and its realization by means of dual frames.
3. Study the **local** properties of the involved frame elements; this is of particular importance in applications, when frame coefficients are modified.
Introducing Flexibility in time

Regular Gabor frames:

\[ g_{n,k}(t) = M_{kb} T_{nag}(t), \quad n, k \in \mathbb{Z} \]

Nonstationary Gabor frames:

\[ g_{n,k}(t) = M_{kb_n} g_n(t), \quad n, k \in \mathbb{Z} \]
Introducing Flexibility in Frequency

Regular Gabor frames:

\[ g_{n,k}(t) = M_{kb} T_{nag}(t), \quad n, k \in \mathbb{Z} \]

Nonstationary Gabor frames:

\[ g_{n,k}(t) = T_{na_k} g_k(t) = g_k(t - na_k), \quad n, k \in \mathbb{Z} \]
Introducing Flexibility

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- Time-frequency (TF) analysis
  - Local aspects
  - Global aspects
- Adaptive representations
  - Nonstationary and Quilted Gabor Frames
- Localization Operators
  - Spectral Properties
  - Gabor multipliers
- Applications
  - Nonstationary Gabor frames
  - Sparsity
- Summary

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Local and Global Aspects of Time-Frequency Analysis
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Dörfler, Monika
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Existence of Nonstationary Gabor Frames

Gabor analysis: *Walnut representation of frame operator* as central tool to existence results. For nonstationary Gabor frames:

### Frame operator

Given a frame $\{g_j, j \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$, the *frame operator* is given as the operator $Sf = \sum_j \langle f, g_j \rangle g_j$ on $L^2(\mathbb{R})$.

### Walnut representation for nonstationary Gabor frames (MD/Matusiak 2012)

The frame operator $S$ corresponding to a nonstationary Gabor frame admits a Walnut representation for $f \in L^2(\mathbb{R})$:

$$Sf = \sum_{k,n \in \mathbb{Z}} b_n^{-1} g_n(t - kb_n^{-1})g_n(t) \cdot T_{kb_n^{-1}}f.$$
For compactly supported windows with sufficient overlap, the frame operator is diagonal if the frequency sampling constants $b_n$ are sufficiently small:

$$Sf(t) = \left( \sum_n \frac{1}{b_n} |g_n(t)|^2 \right) f(t).$$

This situation is called, alluding to the famous ”painless non-orthogonal expansions”, the **painless case**.
Existence of Nonstationary Gabor Frames

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This situation is called, alluding to the famous "painless non-orthogonal expansions", the **painless case**. For polynomially decaying window an existence result in parallel to the regular case can be obtained.
Existence of Nonstationary Gabor Frames

Let \( g = \{g_n \in W(L^\infty, \ell^1) : k \in \mathbb{Z}\} \) be a set of windows such that

- for some positive constants \( A_0, B_0 \):
  \[
  0 < A_0 \leq \sum_{n \in \mathbb{Z}} |g_n(t)|^2 \leq B_0 < \infty \quad \text{a.e. ;}
  \]
- for all \( n \in \mathbb{Z} \), the windows decay polynomially around a \( \delta \)-separated set \( \{a_n : n \in \mathbb{Z}\} \) of time-sampling points \( a_n \)

\[
|g_n(t)| \leq C_n(1 + |t - a_n|)^{-p_n}
\]

where \( p_n \in [p_L, p_U] \subset \mathbb{R}, p_L > 2 \) and \( C_n \in [C_L, C_U] \).

Then there exists a sequence \( \{b_n^0\}_{n \in \mathbb{Z}} \), such that for \( b_n \leq b_n^0, n \in \mathbb{Z} \), the nonstationary Gabor system \( \mathcal{G}(g, b) \) forms a frame for \( L^2(\mathbb{R}) \).
In the painless case, convenient reconstruction is possible (→ applications!), since:

\[ f = S^{-1} S f = \sum_{n,k \in \mathbb{Z}} \langle f, M_{kb_n} g_n \rangle M_{kb_n} \frac{g_n}{\sum_{n'} \frac{1}{b_{n'}} |g_{n'}|^2}. \]


Flexibility in both time and frequency, with local structure, is modeled by Quilted Frames. (MD, Quilted Gabor frames - A new concept for adaptive time-frequency representation, 2011.) Here, no ”painless case” is possible...
Localization in Time-Frequency

Basic idea in all presented time-frequency frames was to compute in some way the "local" energy of $f$ in a small region of the time-frequency plane, i.e.

$$f \rightarrow \langle f, g_{n,k} \rangle,$$

hoping, that the coefficients $\langle f, g_{n,k} \rangle$, $n, k \in \mathbb{Z} \times \mathbb{Z}$ globally represent $f$ in an appropriate way and characterize the function space to which $f$ is assumed to belong.
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More general approach to TF-localization: use an operator other than the simple projection $f \mapsto \langle f, g_{n,k} \rangle g_{n,k}$.

E.g. by restriction of the STFT prior to reconstruction: $\rightarrow$ Localization operators
TF Localization operator

Recall: Set $\pi(z)g(t) = g(t - x)e^{2\pi i \xi t}$, for some $g \in L^2(\mathbb{R})$ with $\|g\|_2 = 1$, then

$$f = \int_{\mathbb{R}^2} \mathcal{V}_g f(z) \pi(z) g \, dz.$$
Localization in Time-Frequency

**TF Localization operator**

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$$f = \int_{\mathbb{R}^2} V_g f(z) \pi(z) g \, dz.$$  

Let $\sigma \in L^1(\mathbb{R}^2)$, then the localization operator $H_{\sigma, g}$ is defined by

$$H_{\sigma, g} f = \int_{\mathbb{R}^2} \sigma(z) V_g f(z) \pi(z) g \, dz = V_g^* \sigma V_g f.$$
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For real symbols $\sigma$: $H_{\sigma, g}$ self-adjoint; compact (even trace-class).
Exploiting the spectral decomposition of Localization operators, it is possible to show that modulation spaces can be characterized by a family of localization operators $H_{T,\lambda,\sigma}$, for $\lambda \in \Lambda$, (MD, Gröchenig: *Time-frequency partitions and characterizations of modulations spaces with localization operators*, 2011). Requirement: $\sum_{\lambda \in \Lambda} T_{\lambda,\sigma} \approx 1$.

Using a generalization of this result by J. Romero (2012), it was, reciprocally, shown, that the union of a finite number of eigenfunctions of the localization operators corresponding to an irregular cover, also provide a frame and thus to a characterization of the same function spaces. (MD, Romero: *Frames adapted to a phase-space cover*, 2012).

What do we know about the eigenfunctions?
Daubechies (1988) considered \( g(t) = \varphi(t) = 2^\frac{1}{4} e^{-\pi t^2} \) and \( \sigma(z) = \chi_\Omega(z) \) and the eigenvalue problem

\[
H_\Omega f := H_{\chi_\Omega, \varphi} f = \lambda f
\]

for \( \Omega \) a disc centered at zero.

Then, the eigenfunctions of \( H_{\chi_\Omega, \varphi} \) are the Hermite functions.
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$$H_{\Omega} f := H_{\chi_{\Omega}, \varphi} f = \lambda f$$

(2)

for $\Omega$ a disc centered at zero. Then, the eigenfunctions of $H_{\chi_{\Omega}, \varphi}$ are the Hermite functions. Solutions are functions with best concentration in $\Omega$ in the sense

$$C_{\Omega}(f) = \frac{\int_{\Omega} |\mathcal{V}_\varphi f(z)|^2 dz}{\|f\|_2^2}.$$ 

(3)
Consider the "inverse problem":
Given a localization operator $H_{\Omega}$ with unknown localization domain $\Omega$, can we recover the shape of $\Omega$ from information about its eigenfunctions and eigenvalues?
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**Theorem (Abreu,MD, 2012)**

Let $\Omega \subset \mathbb{R}^2$ be simply connected. If one of the eigenfunctions of the localization operator $H_\Omega$ is a Hermite function, then $\Omega$ must be a disk centered at 0.
Main ideas to proof:

- Eigenfunctions (resp. their STFT) of localization operators are doubly orthogonal.
- The Bargmann transform maps eigenvalue-problem to the Bargmann-Fock space of analytic functions, with Hermite functions being mapped to the (appropriately normalized) monomials.
- The double orthogonality of the monomials with respect to a simply connected domain $\Omega$ and a concentric measure forces $\Omega$ to be a disk (at zero).
**Remark:** Eigenvalue problem for TF localization operators may be set in the more general context of restricting reproducing formulas:

\[
\int_{D_R} F(z)K(\bar{z}, w)\,d\mu(z) = \lambda F(w). \tag{4}
\]

\[
d\mu(z) = e^{-\pi|z|^2}\,dz \quad \rightarrow \quad \text{Gabor localization problem}
\]

\[
d\mu(z) = (1 - |z|^2)^\alpha\,dz \quad \rightarrow \quad \text{wavelet localization}.
\]
Gabor multipliers are the discrete version of TF localization operators - Important in applications, ubiquitously used.
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Gabor multipliers

Gabor multipliers are the discrete version of TF localization operators - Important in applications, ubiquitously used. Here, the TF-localization process is defined via the *discrete* TF-representation provided by frames. Consider Gabor frames with windows \( \varphi, \phi \) and lattice \( \Lambda \) and the associated analysis operator \( C_{\varphi,\Lambda}(f) = \langle f, \pi(\lambda)\varphi \rangle, \lambda \in \Lambda, \) and synthesis operator \( C_{\phi,\Lambda}^* \).

**Definition: Gabor Multiplier**

Let \( m \cdot C_{g,\Lambda} f \) denote pointwise multiplication by \( m \in \ell^\infty(\Lambda) \). Then, a Gabor multiplier is defined as

\[
G_m : f \in \mathcal{H} \mapsto G_m f = C_{\phi,\Lambda}^*(m \cdot C_{\varphi,\Lambda} f).
\]
Gabor multipliers

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Discretization $\rightarrow$ aliasing effects studied in Fourier transform domain. For operator sampling: Fourier transform replaced by Plancherel transform $\rightarrow$ spreading representation of operators:

**Spreading representation of operators**

Let a Hilbert-Schmidt operator $H : \mathbb{R} \mapsto \mathbb{R}$ with integral kernel $\kappa_H$ be given. Then, its spreading function $\eta_H(b, \nu) = \int_{\mathbb{R}} \kappa_H(x, x - b) e^{-2\pi i\nu x} dx$ is in $L^2(\mathbb{R}^2)$ and provides the following integral representation for $H$:

$$H = \int_b \int_{\nu} \eta_H(b, \nu) \pi(b, \nu) db d\nu.$$
Approximation with Gabor multipliers

MD/Torresani 2010

The spreading function of a Gabor multiplier $G_m$ takes the form

$$\eta_{G_m}(b, \nu) = M(b, \nu) \cdot V_\varphi \phi(b, \nu),$$

where $M$ is the symplectic Fourier transform of $m$. 

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Local and Global Aspects of Time-Frequency Analysis
The spreading function of a Gabor multiplier $G_m$ takes the form

$$\eta_{G_m}(b, \nu) = \mathcal{M}(b, \nu) \cdot V_\phi \phi(b, \nu),$$

where $\mathcal{M}$ is the symplectic Fourier transform of $m$. $m$ is a sequence, defined on a lattice $\Lambda$, hence its (symplectic) two-dimensional Fourier transform is a periodic function with period given by a fundamental domain of the adjoint lattice.
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where $\mathcal{M}$ is the symplectic Fourier transform of $m$.

$m$ is a sequence, defined on a lattice $\Lambda$, hence its (symplectic) two-dimensional Fourier transform is a periodic function with period given by a fundamental domain of the adjoint lattice. Consequently, the best approximation of an operator with spreading function $\eta_H$ is obtained via its spreading function by minimizing the error due to aliasing components. Additionally, error estimates can be relatively easily derived.
Approximation with Gabor multipliers

Figure: Aliasing in operator approximation by Gabor multipliers.
Approximation with Gabor multipliers

MD/Torresani 2010

Fix two windows $g$, $h$ and a lattice $\Lambda = b_0 \mathbb{Z} \times \nu_0 \mathbb{Z}$. The best Gabor multiplier approximation of a Hilbert-Schmidt operator $H$ with spreading function $\eta_H$ is given by the sequence $m$ whose discrete symplectic Fourier transform reads

$$M(b, \nu) = \frac{\sum_{n,k=-\infty}^{\infty} \mathcal{V}_g h \left( b + \frac{n}{\nu_0}, \nu + \frac{k}{b_0} \right) \eta_H \left( b + \frac{n}{\nu_0}, \nu + \frac{k}{b_0} \right)}{\sum_{n,k=-\infty}^{\infty} \left| \mathcal{V}_g h \left( b + \frac{n}{\nu_0}, \nu + \frac{k}{b_0} \right) \right|^2}$$

Proof is based on a variational argument.
Approximation with Gabor multipliers

For a given lattice $\Lambda = b_0 \mathbb{Z} \times \nu_0 \mathbb{Z}$, let $\Lambda^\circ = t_0 \mathbb{Z} \times \xi_0 \mathbb{Z}$ with $t_0 = 1/\nu_0$, $\xi_0 = 1/b_0$, and $\Pi^\circ f(\zeta) = \sum_{\lambda^\circ \in \Lambda^\circ} f(\zeta + \lambda^\circ)$, $\zeta \in \Omega^\circ$.

MD/Torresani 2011

Let $H_\sigma$ be localization operator with respect to analysis and synthesis windows $g$ and $h$, respectively. Denote by $\tilde{\sigma}$ the symplectic Fourier transform of $\sigma$ and by $T' = G_m$ the best Gabor multiplier approximation with the same windows, and lattice $\Lambda$. Then, the approximation error is given by

$$
\|H_\sigma - T'\|_H^2 = \int_{\Omega^\circ} \left[ \Pi^\circ((\tilde{\sigma} \cdot V_g h)^2)(\zeta) - \frac{\Pi^\circ(\tilde{\sigma} \cdot |V_g h|^2)(\zeta)^2}{\Pi^\circ(|V_g h(\zeta)|^2)} \right] d\zeta
$$
Application of nonstationary Gabor frames

- constant Q transform (CQT): method of transforming a time-domain signal $f$ to the time-frequency domain such that the ratio of the center frequencies to the respective bandwidth (Q-factor) is fixed.


- well-suited for the analysis of music signals because of the geometric spacing of the center frequencies.

- Perfect invertibility and real-time implementation has only been provided by NSG-implementation. (Holighaus, MD, Velasco, Grill A framework for invertible, real-time constant-Q transforms IEEE Trans. Audio Speech Lang. Process. 21, 4 (2013))
Application of nonstationary Gabor frames

Idea of geometric spacing in Frequency:

- Linear frequency spacing
- Fixed time-frequency resolution
- Geometric frequency spacing
- Fixed center frequency to resolution ratio (Q-factor)
Application of nonstationary Gabor frames

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Sparsity
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Figure: STFT with adapted window
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Local and Global Aspects of Time-Frequency Analysis
Computation Time vs. Signal Length

- CQT
- CQ-NSGT (no primes)
- CQ-NSGT (including primes)

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Computation Time vs. Number of Bins per Octave

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<td>96</td>
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Sparsity and structured sparsity

- Additional idea: search for those coefficients, which best correspond to some prior knowledge.
- By choice of frame (dictionary), only few components for a good representation are needed.
- Minimize $\Delta(f) = \| \sum_{n,k} c_{n,k} \varphi_{n,k} - \hat{f} \|_2^2 + \mu \| c \|_{\ell^1}$ ...and obtain...
Sparsity and structured sparsity

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Local and Global Aspects of Time-Frequency Analysis
Sparsity and structured sparsity
Sparsity and structured sparsity

Problems with LASSO (pure thresholding): structures not considered.
Sparsity and structured sparsity

Modeling Structures/Persistence:
Sparsity and structured sparsity

The signal
Application to multi-layer decomposition.
Application to multi-layer decomposition.
Application to multi-layer decomposition.

- Listen to residual!

\[ \text{(Reconstruction Ton+Trans)} \]
Summary and Perspectives

- Time-frequency analysis requires careful consideration of local and global aspects in order to meet important issues such as invertibility, satisfactory time- and frequency resolution and numerical feasibility.

- Flexible and adaptive transforms involve deep mathematical issues and are useful in applications.

- Sparsity is a powerful concept and leads to exciting mathematical problems.

- Discretization involves additional interesting insight and challenges and is closely related to classical sampling theory.
Summary and Perspectives

Some open questions/new directions:

- Construction of flexible frames in dependence on information criteria, approximate dual frames and error estimates for reconstruction.

- Investigate properties of eigenfunctions of general localization operators; sampling of time-frequency localized functions.

- Relating computational results to corresponding continuous problems.

- Structured sparsity: convergence of iterative algorithms; applications: inpainting, novelty detection, source separation.
Thank you for your attention!

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Sparsity and structured sparsity

Details:

Figure: Generalization: $\Omega$ is an annulus.
In the Gabor case, the choice of the Gaussian function \( \varphi(t) = 2^{\frac{1}{4}} e^{-\pi t^2} \) allows the translation of the time-frequency localization operator \( H_{\chi, \varphi} \) to the complex analysis set-up via the **Bargmann transform** \( \mathcal{B} \):

\[
\mathcal{B} f(z) = \int_{\mathbb{R}} f(t) e^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2} dt = e^{-i\pi x \xi + \pi \frac{|z|^2}{2}} \mathcal{V}_\varphi f(x, -\xi).
\]  

(7)
In the Gabor case, the choice of the Gaussian function \( \varphi(t) = 2^{\frac{1}{4}} e^{-\pi t^2} \) allows the translation of the time-frequency localization operator \( H_{\chi,\varphi} \) to the complex analysis set-up via the Bargmann transform \( \mathcal{B} \):

\[
\mathcal{B} f(z) = \int_{\mathbb{R}} f(t) e^{2\pi i z t - \pi t^2 - \frac{\pi}{2} z^2} \, dt = e^{-i\pi x \xi + \pi |z|^2} \mathcal{V}_\varphi f(x, -\xi).
\] (7)

\( \mathcal{B} \) maps \( L^2(\mathbb{R}) \) unitarily onto \( \mathcal{F}^2(\mathbb{C}) \), the Bargmann-Fock space of analytic functions with the inner product obtained by choosing the measure \( d\mu(z) = e^{-\pi |z|^2} \, dz \).
Let $h_n(t) = c_n e^{\pi t^2} \left( \frac{d}{dt} \right)^n (e^{-2\pi t^2})$ be the Hermite functions. The normalized monomials

$$e_n = (\pi^n / n!) \cdot z^n = B h_n(z) = e^{-i \pi x \xi + \pi |z|^2} \mathcal{V}_\phi h_n(z)$$

form an orthonormal basis for $\mathcal{F}^2(\mathbb{C})$. 

**Eigenfunctions of Localization operators**
Let \( h_n(t) = c_n e^{\pi t^2} \left( \frac{d}{dt} \right)^n (e^{-2\pi t^2}) \) be the Hermite functions. The normalized monomials

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e_n = (\pi^n / n!) \cdot z^n = B h_n(z) = e^{-i\pi x \xi + \pi |z|^2} \mathcal{V}_\varphi h_n(z)
\]

form an orthonormal basis for \( \mathcal{F}^2(\mathbb{C}) \).

As a direct consequence of the unitarity of \( B \) and \( \mathcal{V}_\varphi \), the set \( \{ \mathcal{V}_\varphi h_n, n \in \mathbb{N} \} \) is orthogonal over all discs \( D_R \).
Eigenfunctions of Localization operators

By unitarity of the Bargmann transform, problem $H_\Omega f = \lambda f$ is equivalent to

$$\int_\Omega \mathcal{V}_\varphi f(z) B(\pi(z)\varphi)(w) \, dz = \lambda Bf(w)$$

$B(\pi(z)\varphi)(w) = e^{-\pi ix\xi} e^{-\pi |z|^2/2} e^{\pi w\bar{z}}$, hence

$$\int_\Omega Bf(z) e^{\pi \bar{z}w - \pi |z|^2} \, dz = \lambda Bf(w)$$

and the eigenvalue problem on $L^2(\mathbb{R})$ is equivalent to

$$\int_\Omega F(z) e^{\pi \bar{z}w - \pi |z|^2} \, dz = \lambda F(w)$$

on $\mathcal{F}^2(\mathbb{C})$. 
Expand $e^{\pi \bar{z} w}$ in its power series to obtain

$$
\lambda F(w) = \sum_{n=0}^{\infty} \frac{\pi^n}{n!} w^n \int_\Omega F(z) \bar{z}^n e^{-\pi |z|^2} \, dz
$$  \hspace{1cm} (8)

By assumption, one $z^m$ solves (8) for $\lambda = \lambda_m$, hence, setting $F(z) = z^m$ gives

$$
\lambda_m w^m = \sum_{n=0}^{\infty} \frac{\pi^n}{n!} w^n \int_\Omega z^m \bar{z}^n e^{-\pi |z|^2} \, dz
$$

which implies

$$
\int_\Omega \bar{z}^n z^m e^{-\pi |z|^2} \, dz = \lambda_m \frac{m!}{\pi^m} \delta_{n,m}.
$$
Eigenfunctions of Localization operators

From
\[ \int_{\Omega} z^n z^m e^{-\pi |z|^2} \, dz = \lambda_m \frac{m!}{\pi^m} \delta_{n,m} \]
and setting \( n = m + k \) leads to
\[ \int_{\Omega} |z|^{2m} \overline{z}^k e^{-\pi |z|^2} \, dz = \lambda \delta_{k,0}, \text{ for all } k \geq 1 \tag{9} \]
such that the results on the localization domain of monomials can be applied to conclude that \( \Omega \) must be a disk centered at zero.