Construction of frames by discretization of phase space

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Section 1

Introduction
Fourier Transform

Decomposes a signal $f$ into pure frequencies.

Let $f \in L^1(\mathbb{R})$, then its Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{2\pi i \xi t} \, dt$$

(1)

Left: signal, Right: absolute values of the Fourier Transform
Time-frequency analysis

Two steps:

1. Windowing of the signal to get time localization
2. Fourier Transform to extract the local frequencies

Left: windowed signal, Right: absolute value of the FT
Windowed Fourier Transform

Modulation, Translation

Let \( f \in L^2(\mathbb{R}) \) and \( x, \xi \in \mathbb{R} \), then translation, modulation and TF-shifts are defined as

\[
T_x f(t) = f(t - x), \quad M_\xi f(t) = e^{2\pi i \xi t} f(t), \quad \pi(x, \xi) = M_\xi T_x
\]  \hspace{1cm} (2)

STFT

Given a window function \( g \in L^2(\mathbb{R}) \) and a signal \( f \in L^2(\mathbb{R}) \). Then the Short-Time Fourier Transform of \( f \) with respect to \( g \) is is defined as

\[
V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \mathcal{F}(f \cdot T_x \bar{g})(\xi), \quad \text{for } (x, \xi) \in \mathbb{R}^2.
\]  \hspace{1cm} (3)
**TF-plane:** \( \mathbb{R}^2 \), \( \lambda = (x, \xi) \) corresponds to time-frequency position

**Gabor Transform:** sampling of the STFT on the TF-plane (phase space)

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**Gabor Transform**

Let \( \Lambda \) be a discrete subset of the TF-plane and \( f, g \in L^2(\mathbb{R}) \). Then

- \( \mathcal{G}(g, \Lambda) = \{ \pi(\lambda) g \}_{\lambda \in \Lambda} \) is called *Gabor family*,
- the transform coefficients are given by \( c_{\lambda} = \langle f, \pi(\lambda) g \rangle \), \( \lambda \in \Lambda \).
Gabor Transform (discrete)

For signals of length $L$, i.e. $f \in \mathbb{C}^L$:
- $T_x$, $M_\xi$ for $x, \xi \in \mathbb{Z}_L$
- TF-plane is given by $\mathbb{Z}_L^2$
- window function $g \in \mathbb{C}^L$
Frames are a generalization of bases and allow for redundancy.

Frame

Given a Hilbert space $\mathcal{H}$, then we call a countable family $\{g_i\}_{i \in I}$, a frame if it satisfies for some $0 < A \leq B < \infty$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, g_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{H}$$  \hspace{1cm} (4)

Theorem

Let $\{g_i\}_{i \in I}$ be a frame for the Hilbert space $\mathcal{H}$. Then any $f \in \mathcal{H}$ can be reconstructed (in a stable way) from its transform coefficients $\{\langle f, g_i \rangle\}_{i \in I}$. 

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Reconstruction from transform coefficients: via inversion of the frame operator

\[ S : \mathcal{H} \rightarrow \mathcal{H}, \ Sf = \sum_{i \in I} \langle f, g_i \rangle g_i. \quad (5) \]

For any given signal \( f \in \mathcal{H} \) this leads to the reconstruction formula

\[ f = \sum_{i \in I} \langle f, g_i \rangle S^{-1} g_i. \quad (6) \]
Gabor frames

Gabor frames are especially attractive when one samples on a discrete subgroup $\Lambda$ of the TF-plane, i.e. a lattice.

Theorem

Let $\mathcal{G}(g, \Lambda)$ for $g \in L^2(\mathbb{R})(g \in \mathbb{C}^L)$ be a Gabor frame and $\Lambda \leq \mathbb{R}^2$ ($\Lambda \leq \mathbb{Z}^2_L$) a lattice. Then

$$ f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\tilde{g}, \quad \text{with } \tilde{g} = S_{g,\Lambda}^{-1}g. \quad (7) $$

The above theorem does NOT hold true in general when $\Lambda$ is not a lattice.
Section 2

Finite discrete Gabor transform on general lattices
Discrete Gabor Transform on general lattices

- fast computational algorithms available for separable lattices
- investigation of subgroups of $\mathbb{Z}_L^2$
- use of metaplectic operators to reduce the computations to rectangular case
- our fast algorithms are implemented in the open source toolbox LTFAT (in Matlab and C)
The finite discrete TF-plane is given by $\mathbb{Z}_L \times \mathbb{Z}_L = \mathbb{Z}_L^2$.
Subgroups of $\mathbb{Z}_L^2$

**Theorem (Hampejs, Holighaus, Tóth, W.)**

Let $L \in \mathbb{N}$ and define

$$I_L := \left\{ (a, b, t) \in \mathbb{N}^2 \times \mathbb{N}_0 : a, b \mid L, 0 \leq t \leq \gcd(L/a, b) - 1 \right\}. \quad (8)$$

For $(a, b, t) \in I_L$ define

$$\Lambda_{a,b,t} := \left\{ (ia, itb/\gcd(L/a, b) + jb)^T : \right.$$

$$\left. 0 \leq i \leq L/a - 1, 0 \leq j \leq L/b - 1 \right\}, \quad (9)$$

then $\Lambda_{a,b,t}$ is a subgroup of order $\frac{L^2}{ab}$ of $\mathbb{Z}_L^2$ and the map $(a, b, t) \mapsto \Lambda_{a,b,t}$ is a bijection between the set $I_L$ and the set of subgroups of $\mathbb{Z}_L^2$. 
Subgroups of $\mathbb{Z}_L^2$

This theorem allows to

- write any subgroup of $\mathbb{Z}_L^2$ in the form

$$\Lambda = \begin{pmatrix} a & 0 \\ s & b \end{pmatrix} \mathbb{Z}_L^2,$$

for uniquely determined $a$, $b$ and $s$.
- count the number of subgroups.
- count the number of subgroups of a given order.
Symplectic matrices: deformations of the lattice

Metaplectic operator: corresponding operation on the signal level

**Theorem (Feichtinger et. al)**

For any matrix $A \in \mathbb{Z}_L^{2 \times 2}$ with $\det A = 1$, there exists a metaplectic operator $U_A$ and a phase factor $\phi_A$, such that for all $\lambda \in \mathbb{Z}_L^{2}$

$$U_A\pi(\lambda) = \phi_A(\lambda)\pi(A\lambda)U_A.$$  \hfill (11)

Explicit symplectic/metaplectic pairs:

$$F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto U_F = \mathcal{F}$$

$$S_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mapsto U_{S_c} = \left( f(\cdot) \mapsto f(\cdot) \exp(\pi ic \cdot^2 (L + 1)/L) \right).$$
We use metaplectic operators to perform computations on non-separable lattices through known algorithms.

Theorem (W., Holighaus, Søndergaard)

Let $A \in \mathbb{Z}_L^{2 \times 2}$. There exist $s_0, s_1 \in \mathbb{Z}_L$ and $V \in \mathbb{Z}_L^{2 \times 2}$ with $|\det(V)| = 1$, such that

$$A = P_{s_0,s_1} DV,$$

(12)

where $D \in \mathbb{Z}_L^{2 \times 2}$ is diagonal and

$$P_{s_0,s_1} = S_{-s_1} F^{-1} S_{s_0} F.$$

(13)
Section 3

(Optimal) construction of Gabor systems
Adaption of the lattice

Adjoint lattice

Let $\Lambda$ be a lattice in $\mathbb{R}^2$. Then the adjoint lattice is given by

$$\Lambda^0 = \{ \lambda^0 : \pi(\lambda)\pi(\lambda^0) = \pi(\lambda^0)\pi(\lambda) \text{ for all } \lambda \in \Lambda \}.$$  \hspace{1cm} (14)

Theorem (Janssen test)

Let $g \in S_0$, $\|g\|_2 = 1$ and $\Lambda$ be a lattice. Assume that

$$\sum_{\lambda^0 \in \Lambda^0} |V_{gg}(\lambda^0)| \leq 1 + D < 2.$$  \hspace{1cm} (15)

Then $G(g, \Lambda)$ is a frame with frame bounds satisfying

$$\text{vol}(\Lambda)^{-1}(1 - D) \leq A \leq B \leq \text{vol}(\Lambda)^{-1}(1 + D).$$  \hspace{1cm} (16)
Adaption of the lattice II

**Janssen test**: sufficient condition for frames.

**Periodization of** $V_g g$: necessary condition for frames.

**Proposition (W.)**

Let $\|g\|_2 = 1$ and $\Lambda$ be a lattice and define the function

$$F_{g,\Lambda} = \sum_{\lambda \in \Lambda} |V_g g(x - \lambda)|^2.$$  \hspace{1cm} (17)

Then

$$A \leq \min_x F_{g,\Lambda}(x) \leq \max_x F_{g,\Lambda}(x) \leq B.$$  \hspace{1cm} (18)

If $F_{g,\Lambda}$ is constant on the TF-plane then $\mathcal{G}(g, \Lambda)$ is tight.
Frame bound estimates

Optimal sampling for a finite discrete Gaussian window \((L = 420)\)
We are interested in a solution of the problem

\[ g_{\text{opt}} = \arg \max_{\|g\|_{L^2(\mathbb{R})} = 1} \int_{\mathbb{R}^2} m(\lambda) |V_g g(\lambda)|^2 \, d\lambda. \quad (19) \]

Define the concentration measure for \( g, h \in L^2(\mathbb{R}) \)

\[ \Gamma(g, h) = \int_{\mathbb{R}^2} m(\lambda) |V_h g(\lambda)|^2 \, d\lambda. \quad (20) \]
Iterative Algorithm

Starting window: $h_0 \in L^2(\mathbb{R})$

\[
\begin{align*}
    h_1 &= \arg \max_{\|g\|=1} \Gamma(g, h_0) \\
    h_2 &= \arg \max_{\|g\|=1} \Gamma(g, h_1) \\
    &\vdots
\end{align*}
\]

If this procedure converges we call the limit $g_{opt}$. 
Proposition (Feichtinger, Onchis, Ricaud, Tórresani, W.)

Assume that the algorithm converges for any starting point $h_0 \in B$, where $B$ is a subset of the unit ball in $L^2(\mathbb{R})$. Then the limiting function is the unique solution of the problem

$$g_{\text{opt}} = \arg \max_{g \in B} \int_{\mathbb{R}^2} m(\lambda) |V_g \lambda|^2 \, d\lambda.$$  \hfill (21)

Proposition (Feichtinger, Onchis, Ricaud, Tórresani, W.)

Let the weight function $m(\lambda)$ be positive, bounded and even with respect to $\lambda$. Then the sequence of numbers $a_k = \Gamma(h_{k+1}, h_k)$ is convergent.
Finite discrete algorithm

Top: two different masks, Bottom: $V_{gg}$ of the optimizing functions
Section 4

Warped time-frequency frames
Filter banks for $L^2(D)$

$D$ is either $\mathbb{R}$ or $\mathbb{R}^+$ in our examples.

**Filter bank**

A filter bank consists of

- a family of filters $g = \{g_m\}_{m \in \mathbb{Z}}, \ g_m \in L^2(D)$
- downsampling factors $a = \{a_m\}_{m \in \mathbb{Z}}, \ a_m \in \mathbb{R}^+$

The filter bank is then given by the functions

$$g_{m,n} = T_{n a_m} \tilde{g}_m.$$

(22)

A Gabor family is a filter bank with the choice $g_m = T_{mb}g$ and $a_m = a$ for some window function $g \in L^2(\mathbb{R})$ and time and frequency step $a, b \in \mathbb{R}^+$. 

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Warping

Warping function

Warping is defined by a function $F : D \to \mathbb{R}$, such that

- $D$ is an interval (here always $\mathbb{R}$ or $\mathbb{R}^+$)
- $F$ is bijective
- $F \in C^1$ and $F' > 0$
- $w(t) = (F^{-1})' (t)$ is $\nu$-moderate ($\nu$ submultiplicative)

Warped filter bank

Let $F$ be a warping function, $\theta : \mathbb{R} \to \mathbb{R}$ be an (appropriate) window and $a$ downsampling factors. Then $G(\theta, F, a)$ has the filter functions

$$g_m(t) = \sqrt{a_m} \theta(F(t) - m) = \sqrt{a_m} (T_m \theta) \circ F(t)$$

(23)
Log warping

Left: regular translates $T_m \theta$, Right: log-warped wavelet filters $(T_m \theta) \circ \log$
Proposition (Holighaus, W.)

Let $G(\theta, F, a)$ such that $\text{supp}(\theta) \subseteq [c, d]$ for some constants $0 < c < d < \infty$. If $a_m^{-1} \geq F^{-1}(d + m) - F^{-1}(c + m)$ for all $m$, then $G(\theta, F, a)$ forms a frame with frame bounds $A$ and $B$, if and only if

$$0 < A \leq \sum_{m \in \mathbb{Z}} |T_m \theta|^2 \leq B < \infty.$$  \hfill (24)

In that case, the canonical dual frame is of the form $G(\tilde{\theta}, F, a)$, with

$$\tilde{\theta}(t) = \frac{\theta(t)}{\sum_m |T_m \theta(t)|^2}, \text{ a.e.}$$  \hfill (25)
Constructions of partitions of unity

Theorem (Shuman, W., Holighaus, Vandergheynst)

Let $K \in \mathbb{N}$ and $a_k \in \mathbb{R}$ for $k \in \{0, 1, \ldots, K\}$, and define

$$q(t) := \sum_{k=0}^{K} a_k \cos \left( 2\pi k \left( t - \frac{1}{2} \right) \right) 1_{\{0 \leq t < 1\}}.$$ (26)

Then for any $R > 2K$

$$\sum_{m \in \mathbb{Z}} \left| q \left( t - \frac{m}{R} \right) \right|^2 = R a_0^2 + \frac{R}{2} \sum_{k=1}^{K} a_k^2, \quad \forall t \in \mathbb{R};$$ (27)

i.e. the squares of a system of regular translates sums up to a constant function.
Different window functions with different degrees of differentiability
Given a warping function $F$.

- Construct some $\theta$ with compact support, such that

$$\sum_{m \in \mathbb{Z}} |T_m \theta|^2 = \text{const}.$$  \hfill (28)

- This is inherited by the warped system

$$\sum_{m \in \mathbb{Z}} |(T_m \theta) \circ F|^2 = \text{const}.$$  \hfill (29)

- Choose the downsampling factors small enough for painless sampling.

- This yields functions $g_{m,n}$ forming a tight frame.
Interesting warping functions

- $F(t) = \log(t)$: leads to wavelets
- $F(t) = \text{sign}(t) c_1 \log \left(1 + \frac{|t|}{c_2}\right)$ (for $c_1, c_2 > 0$): This is the ERB-scale and yields filters that are adapted to the human ear \rightarrow potential applications in audio signal processing.
- $F(t) = \text{sign}(t)(|t| + 1)^r - 1$ (for $r \in (0, 1]$): Can be seen as an interpolation between an ERB-like warping function and the identity (Gabor)
An audio example

Left: standard Gabor transform, Right: warped transform based on ERB-scale (both tight)
A method for optimizing the ambiguity function concentration.

M. Hampejs, N. Holighaus, L. Tóth, and C. Wiesmeyr.
On the subgroups of the group $Z_m \times Z_n$.

C. Wiesmeyr, N. Holighaus, and P. Sondergaard.
Efficient algorithms for discrete Gabor transforms on a nonseparable lattice.

D. Shuman, C. Wiesmeyr, N. Holighaus, and P. Vandergheynst.
Spectrum-Adapted Tight Graph Wavelet and Vertex-Frequency Frames.

An optimally concentrated Gabor transform for localized time-frequency components.

Metaplectic operators on $C^n$.

P. Sondergaard.
Efficient algorithms for the discrete Gabor transform with a long FIR window.