Structured Banach frame decompositions of decomposition spaces

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Applied Functional Analysis Group
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Outline

1. Motivation (3 slides)
2. Structured Banach frame decompositions of decomposition spaces (6 slides)
3. Example applications (5 slides)

Outline

Frames — What are they good for?

Definition

A **frame** is a family $(h_i)_{i \in I}$ in a Hilbert space $\mathcal{H}$ such that

$$\|x\|_{\mathcal{H}} \asymp \left\| (\langle x, h_i \rangle)_{i \in I} \right\|_{\ell^2(I)} \qquad \forall x \in \mathcal{H}. \qquad (*)$$

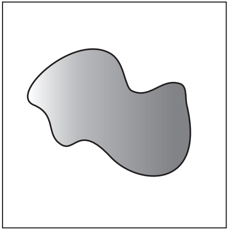
Gröchenig (*Foundations of Time-Frequency Analysis*)

> *"such a characterization can be accomplished with any orthonormal basis [...] with less effort and more simplicity."*

We want more of a nice frame than just property $(*)$.

Properties of nice frames that we consider in this talk

- **Sparse representations of certain types of functions**
  - smooth functions with point singularities (Wavelets)
  - smooth functions with curvilinear singularities (Shearlets, Curvelets)



(image courtesy of Gitta Kutyniok)

Question: **Analysis sparsity** (i.e., $(\langle f, h_i \rangle)_{i \in I}$ is sparse) or **synthesis sparsity** (i.e., $f = \sum c_i h_i$ with $(c_i)_{i \in I}$ sparse)?

- **Characterization of function spaces**
  - Modulation spaces (Gabor frames)
  - Besov and Triebel-Lizorkin spaces (Wavelets)
  - ??? (Shearlets)

The talk in one slide

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We consider **structured systems**, i.e., of the form $\Gamma^{(\delta)} = \left(\gamma^{[i,k,\delta]}\right)_{i \in I, k \in \mathbb{Z}^d}$ with

$$\gamma^{[i,k,\delta]} = |\det T_i|^{\frac{1}{2}} \cdot L_{\delta \cdot T_i^{-T} \cdot k} \left[M_{b_i} \left(\gamma \circ T_i^T \right) \right]$$

for some generator γ and a family of affine maps $(T_i \bullet + b_i)_{i \in I}$.

Examples: Gabor systems, wavelets, shearlets.

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Frequency concentration of structured system: $\gamma^{[i]} := \gamma^{[i,0,\delta]}$ satisfies

$$\mathcal{F}\gamma^{[i]} = |\det T_i|^{-1/2} \cdot L_{b_i}\left(\widehat{\gamma} \circ T_i^{-1}\right).$$

If $\widehat{\gamma}$ is concentrated in $Q \subset \mathbb{R}^d$, then $\mathcal{F}\gamma^{[i]}$ is concentrated in $Q_i := T_i Q + b_i$.
$\rightsquigarrow$ **Structured covering** $\mathcal{Q} = (Q_i)_{i\in I}$.

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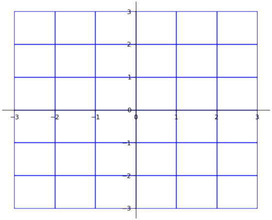
Example: Frequency concentration of **Gabor** systems



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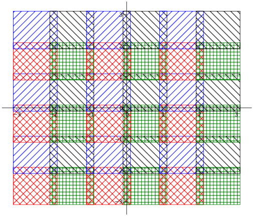

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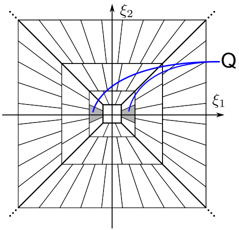
Example: Frequency concentration of **shearlet** systems (image courtesy of Martin Genzel)



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$\rightsquigarrow$ **Structured covering** $\mathcal{Q} = (Q_i)_{i\in I}$.

Covering $\mathcal{Q}$ determines **decomposition space** $\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_w\right)$.

Find **sequence space** $C^{p,q}_w$ and **conditions on** γ (compatible with compact support) s.t.:

$$\mathcal{D}\left(\mathcal{Q}, L^p, \ell^q_w\right) = \left\{ f \ : \ \left(\langle f, \gamma^{[i,k,\delta]}\rangle\right)_{i\in I, k\in\mathbb{Z}^d} \in C^{p,q}_w \right\}$$
$$= \left\{ \sum_{i,k} c^{(i)}_k \cdot \gamma^{[i,k,\delta]} \ : \ \left(c^{(i)}_k\right)_{i\in I, k\in\mathbb{Z}^d} \in C^{p,q}_w \right\},$$

Outline

Decomposition spaces

Consider **structured admissible covering** $\mathcal{Q} = (Q_i)_{i \in I}$ of $\mathbb{R}^d$. This means:

- each Q_i satisfies

$$Q_i = T_i Q + b_i$$

  for fixed open, bounded **base set** $Q \subset \mathbb{R}^d$.
- additional technical conditions (almost always satisfied in practice).

Choose a **regular partition of unity** $\Phi = (\varphi_i)_{i \in I}$ subordinate to $\mathcal{Q}$.

$\varphi_i \in C_c^\infty(Q_i)$, $\sum_{i \in I} \varphi_i \equiv 1$ and some other technical condition.

Choose a **$\mathcal{Q}$-moderate weight** $w = (w_i)_{i \in I}$, i.e., $w_i \leq C \cdot w_j$ if $Q_i \cap Q_j \neq \varnothing$.

For $p, q \in (0, \infty]$, and $g \in \mathcal{R}$, define the **decomposition space (quasi)-norm**

$$\|g\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)} := \left\| \left(w_i \cdot \left\| \mathcal{F}^{-1}(\varphi_i \cdot \widehat{g}) \right\|_{L^p} \right)_{i \in I} \right\|_{\ell^q} \in [0, \infty].$$

The **decomposition space** determined by $\mathcal{Q}, p, q, w$ is

$$\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q) := \left\{ g \in \mathcal{R} \,\middle|\, \|g\|_{\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)} < \infty \right\}.$$

Here $\mathcal{R}$ is a suitable reservoir of distributions (think: $\mathcal{R} = \mathcal{S}'(\mathbb{R}^d)$).

Structured Banach frame decompositions: General idea

If γ has (essential) frequency support in Q, then

$$\gamma^{[i]} := |\det T_i|^{1/2} \cdot M_{b_i} [\gamma \circ T_i^T]$$

has (essential) frequency support in $Q_i = T_i Q + b_i$.

Frequency localization of $(\gamma^{[i]})_{i \in I}$ is compatible with the covering $\mathcal{Q} = (Q_i)_{i \in I}$.

$\rightsquigarrow$ **Hope**: If γ is nice, then

$$f \in \mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$$
$$\iff \left(\left\langle f, \gamma^{[i,k,\delta]} \right\rangle \right)_{i,k} = \left(\left\langle f, L_{\delta \cdot T_i^{-T} k} \, \gamma^{[i]} \right\rangle \right)_{i,k} \text{ decays quickly.}$$

Question: In what sense does γ have to be "nice"?

Note: Even for characterizing L^2, the required "niceness" depends heavily on $\mathcal{Q}$:

- For Gabor systems: Sufficient if γ belongs to the Wiener space.
- For wavelets, γ has to have vanishing moments.

Structured Banach frames — The theorem

We will use the **coefficient space** $C_w^{p,q} = \left\{ c \in \mathbb{C}^{I \times \mathbb{Z}^d} \,:\, \|c\|_{C_w^{p,q}} < \infty \right\}$, where

$$\left\| (c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \right\|_{C_w^{p,q}} = \left\| \left(|\det T_i|^{\frac{1}{2} - \frac{1}{p}} \cdot w_i \cdot \left\| (c_k^{(i)})_{k \in \mathbb{Z}^d} \right\|_{\ell^p} \right)_{i \in I} \right\|_{\ell^q} .$$

Theorem (FV; 2016)

Let $w = (w_i)_{i \in I}$ be $\mathcal{Q}$-moderate and let $p, q \in (0, \infty]$. There are $N \in \mathbb{N}$ and $\sigma, \tau > 0$ (only depending on p, q, d) with the following property:

If $\gamma \in C_c^1(\mathbb{R}^d)$ with $\widehat{\gamma}(\xi) \neq 0$ for all $\xi \in \overline{Q}$, and if

$$\sup_{i \in I} \sum_{j \in I} M_{j,i} < \infty \quad \text{and} \quad \sup_{j \in I} \sum_{i \in I} M_{j,i} < \infty,$$

with

$$M_{j,i} \sim \left(\frac{w_j}{w_i} \right)^{\tau} \cdot \left(1 + \left\| T_j^{-1} T_i \right\| \right)^{\sigma} \left(\fint_{Q_i} \max_{\substack{|\alpha| \leq N \\ |\beta| \leq 1}} \left| [\partial^{\alpha} \widehat{\partial^{\beta} \gamma}] \left(T_j^{-1} (\xi - b_j) \right) \right| \mathrm{d}\xi \right)^{\tau},$$

then there is $\delta_0 > 0$, such that for $0 < \delta \leq \delta_0$, the family $(L_{\delta \cdot T_i^{-T} k}\, \widetilde{\gamma^{[i]}})_{i,k}$, with $\tilde{g}(x) = g(-x)$, forms a Banach frame for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$, with coeff. space $C_w^{p,q}$.

Precise interpretation of the Banach frame property

- The **analysis operator** $A^{(\delta)}$ is bounded, where

$$A^{(\delta)} : \mathcal{D}(\mathcal{Q}, L^p, \ell_w^q) \to C_w^{p,q}, f \mapsto \left([\gamma^{[i]} * f](\delta \cdot T_i^{-T} k) \right)_{i \in I, k \in \mathbb{Z}^d}$$
$$= \left(\left\langle f, \, L_{\delta \cdot T_i^{-T} k} \, \widetilde{\gamma^{[i]}} \right\rangle \right)_{i \in I, k \in \mathbb{Z}^d} .$$

- There is a bounded **reconstruction operator** $R^{(\delta)} : C_w^{p,q} \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$ with

$$f = R^{(\delta)} A^{(\delta)} f \qquad \forall f \in \mathcal{D}(\mathcal{Q}, L^p, \ell_w^q).$$

Structured atomic decompositions

Theorem (FV; 2016)

Under similar conditions on γ as before, there is some $\delta_0 > 0$ such that the family $\left(\gamma^{[i,k,\delta]}\right)_{i,k} = \left(L_{\delta \cdot T_i^{-T} k}\, \gamma^{[i]}\right)_{i \in I, k \in \mathbb{Z}^d}$ is an atomic decomposition of $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$, with coefficient space $C_w^{p,q}$, if $\delta \in (0, \delta_0]$.

Precisely, the **synthesis map**

$$S^{(\delta)} : C_w^{p,q} \to \mathcal{D}(\mathcal{Q}, L^p, \ell_w^q), (c_k^{(i)})_{i,k} \mapsto \sum_{i \in I} \sum_{k \in \mathbb{Z}^d} \left(c_k^{(i)} \cdot \gamma^{[i,k,\delta]}\right)$$

is well-defined and bounded, and we have $f = S^{(\delta)} C^{(\delta)} f$ for all $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$ for some bounded **coefficient map** $C^{(\delta)} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right) \to C_w^{p,q}$.

Observation: For $w_i = |\det T_i|^{\frac{1}{p}-\frac{1}{2}}$, we have $C_w^{p,p} = \ell^p\left(I \times \mathbb{Z}^d\right)$.

Thus: For $0 < p \leq 2$, γ sufficiently nice, and $\delta > 0$ small, we have

$$\begin{aligned}\mathcal{D}(\mathcal{Q}, L^p, \ell_w^p) &= \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \left(\left\langle f, \gamma^{[i,k,\delta,*]}\right\rangle\right)_{i\in I, k\in\mathbb{Z}^d} \in \ell^p(I \times \mathbb{Z}^d) \right\} \\ &= \left\{ \sum_{(i,k)\in I\times\mathbb{Z}^d} c_k^{(i)} \cdot \gamma^{[i,k,\delta]} \,\middle|\, (c_k^{(i)})_{i\in I, k\in\mathbb{Z}^d} \in \ell^p(I \times \mathbb{Z}^d) \right\},\end{aligned}$$

with $\mathcal{Q} = (T_i Q + b_i)_{i\in I}$ and where

$$\gamma^{[i,k,\delta]} = L_{\delta \cdot T_i^{-T} k}\left[M_{b_i}(\gamma \circ T_i^T)\right] \quad \text{and} \quad \gamma^{[i,k,\delta,*]} = L_{\delta \cdot T_i^{-T} k}\left[M_{-b_i}(\widetilde{\gamma} \circ T_i^T)\right].$$

Outline

New proofs of classical results

Corollary

For ***Besov spaces*** $B_s^{p,q}\left(\mathbb{R}^d\right)$*:*

- *"Horrible condition"* $\rightsquigarrow$ *smoothness + localization +* ***vanishing moments****.*
- *Structured system* $\rightsquigarrow$ *Wavelet system.*

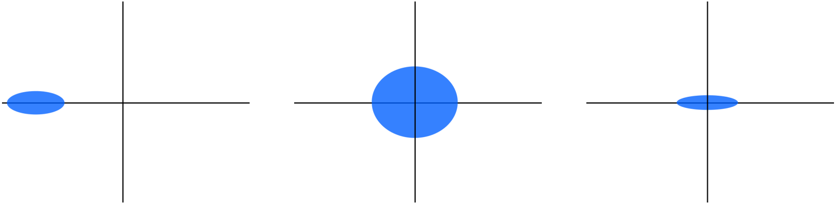
Theorem (FV; 2016)

For α***-modulation spaces*** $M_{s,\alpha}^{p,q}\left(\mathbb{R}^d\right)$*:*

- *"Horrible condition"* $\rightsquigarrow$ *smoothness + localization.*
- *Structured system: Nice structured system (for* $\alpha = 0$*:* ***Gabor system****).*

Shearlets are somewhat similar to wavelets, but use the **parabolic dilations** and **shearing matrices**

$$\begin{pmatrix} 2^{j/2} & 0 \\ 0 & 2^j \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}.$$



A one-minute crash course on shearlets

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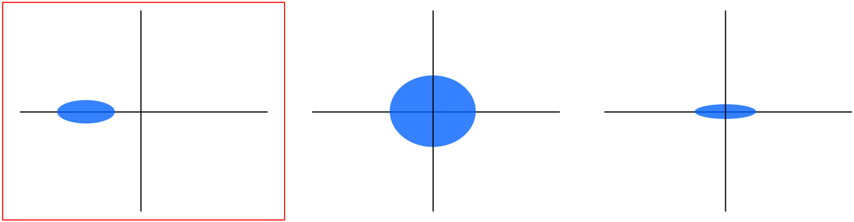

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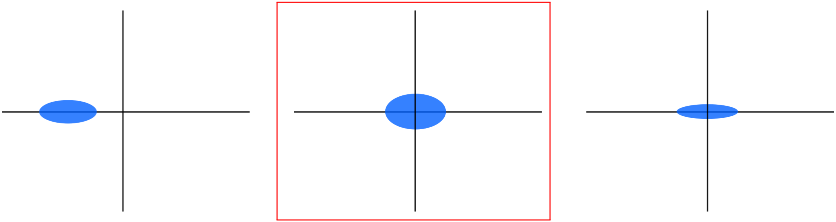

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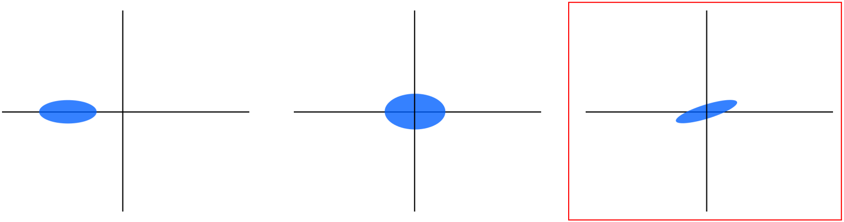

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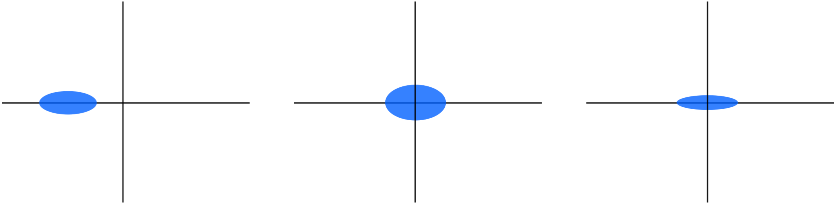
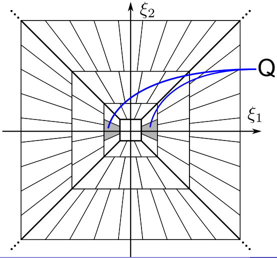
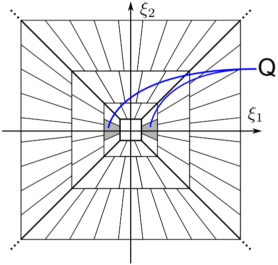


Associated **frequency covering** $\mathcal{S} = (T_i Q + b_i)_{i \in I}$:



Shearlet smoothness spaces

The **shearlet covering** $\mathcal{S} = (T_i Q + b_i)_{i \in I}$:



The spaces $\mathscr{S}_s^{p,q}(\mathbb{R}^2) = \mathcal{D}(\mathcal{S}, L^p, \ell^q_{w^s})$ are called **shearlet smoothness spaces** **(Labate, Mantovani, Negi; 2013)**.

The weight w is chosen such that $w_i \sim 1 + |\xi|$ for $\xi \in Q_i$.

Observation: Structured family generated by ψ is a **cone-adapted shearlet system** **(Guo, Labate, Kutyniok; 2006)**:

$$\Psi^{(\delta)} := \left(\psi^{[i,k,\delta]}\right)_{i \in I, k \in \mathbb{Z}^2} = \mathrm{SH}\left(M_{b_0}(\psi \circ T_0^T), \psi; \delta\right).$$

Theorem (Pein, FV; 2017)

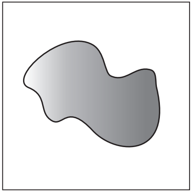
Let $p_0, q_0 \in (0,1]$ and $s_0 \geq 0$. There are $N_1, N_2 \in \mathbb{N}$ such that if $\psi_1, \psi_2 \in C_c^{N_1}(\mathbb{R})$ and $\psi = \psi_1 \otimes \psi_2$ with

$$\widehat{\psi}(\xi) \neq 0 \text{ for } \xi \in \overline{Q} \quad \text{and} \quad \left.\frac{\mathrm{d}^\ell}{\mathrm{d}\,\xi^\ell}\right|_{\xi=0} \widehat{\psi_1}(\xi) = 0 \text{ for } \ell = 0, \ldots, N_2$$

*then there is $\delta_0 > 0$, such that $\Psi^{(\delta)}$ is a Banach frame **and** an atomic decomposition for $\mathscr{S}_s^{p,q}(\mathbb{R}^2)$ for all $p \geq p_0$, $q \geq q_0$, $|s| \leq s_0$, and $0 < \delta \leq \delta_0$.*

Application: Approximation of cartoon-like functions

We consider C^2 **cartoon-like functions** $f \in \mathcal{E}^2$:



(image courtesy of Gitta Kutyniok)

Previously known (Guo, Labate, Lim, Kutyniok et al.): If ψ is a nice mother shearlet, then

- the **analysis coefficients** of f w.r.t. $\mathrm{SH}(\psi;\delta)$ are ℓ^p**-sparse** for $p > \frac{2}{3}$.
- For suitable linear combination f_N of N elements of the **dual shearlet frame**:

$$\|f - f_N\|_{L^2} \leq C_{\delta,\psi} \cdot N^{-1} \cdot (1 + \ln N)^{3/2}.$$

New result (Pein, FV; 2017)

For suitable linear comb. g_N of N elements **of the shearlet frame** $\mathrm{SH}(\psi;\delta)$:

$$\|f - g_N\|_{L^2} \leq C_{\varepsilon,\delta,\psi} \cdot N^{-(1-\varepsilon)} \qquad \forall \varepsilon \in (0,1) \text{ and } N \in \mathbb{N}.$$

Conclusion

In this talk:

- We presented a framework for constructing structured, singly generated, (possibly) compactly supported Banach frame decompositions for the Besov-type decomposition spaces $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$, where $\mathcal{Q} = (T_i Q + b_i)_{i \in I}$.
- We recover known results about wavelet (and Gabor) frames.
- Novel results for cone-adapted shearlets (and α-modulation spaces).

Also possible:

- $Q_i = T_i Q_i' + b_i$, with e.g. $\left\{Q_i' : i \in I\right\}$ finite $\rightsquigarrow$ **finitely generated** frames.

Conclusion of the conclusion

Consider using the **structured Banach frame decomposition framework** if you need to consider sparsity properties of systems of the form

$$\left(L_{\delta \cdot T_i^{-T} k}\left[M_{b_i}\left(\gamma_i \circ T_i^T\right)\right]\right)_{i \in I, k \in \mathbb{Z}^d}.$$

Thank you!

Questions, comments, counterexamples?

☺