Uncertainty and Spectrogram Geometry

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* based on joint work with François Auger and Éric Chassande-Mottin

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Heisenberg (classical)

Second-order (variance-type) measures for the \textbf{individual time and frequency spreadings} of a signal $x(t) \in L^2(\mathbb{R})$ with spectrum $X(\omega)$:

$$\Delta^2_t(x) := \frac{1}{E_x} \int t^2 |x(t)|^2 \, dt$$

$$\Delta^2_\omega(X) := \frac{1}{E_x} \int \omega^2 |X(\omega)|^2 \frac{d\omega}{2\pi}$$

\textbf{Theorem (Weyl, ’27; Gabor, ’46, . . . )}

$$\Delta_t(x) \Delta_\omega(X) \geq \frac{1}{2},$$

\textit{with lower bound attained for Gaussians} $x_*(t) = C e^{\alpha t^2}$, $\alpha < 0$. 
Second-order (variance-type) measure for the \textbf{joint time-frequency spreading} of a signal $x(t) \in L^2(\mathbb{R})$ with energy distribution $C_x(t, \omega; \varphi)$ within Cohen’s class:

$$
\Delta_{t\omega}(C_x) := \frac{1}{E_x} \int \int \left( \frac{t^2}{T^2} + T^2 \omega^2 \right) C_x(t, \omega; \varphi) \, dt \, \frac{d\omega}{2\pi}
$$

\textbf{Theorem (Janssen, ’91)}

- \textit{Wigner} $\Rightarrow$ $\Delta_{t\omega}(W_x) \geq 1$
- \textit{Spectrogram} $\Rightarrow$ $\Delta_{t\omega}(S^h_x) \geq 2$,

\textit{with lower bound attained for Gaussian signals and matched Gaussian windows.}
time-frequency “covariance”

Definition (Cohen, ’95)

\[ c(x) := \int t |x(t)|^2 \frac{d}{dt} \arg x(t) \, dt \]

1. **Interpretation**: covariance quantifies the coupling between time and instantaneous frequency

\[ c(x) = \langle t \omega_x(t) \rangle \]

2. **Intuition**: covariance is zero in case of no coupling

\[ c(x) = \langle t \omega_x(t) \rangle = \langle t \rangle \langle \omega_x(t) \rangle = \langle t \rangle \langle \omega \rangle = 0 \]
From uncertainty... to localization Spectrogram geometry

Schrödinger

Theorem (Schrödinger, ’30)

\[ \Delta_t(x) \Delta_\omega(X) \geq \frac{1}{2} \sqrt{1 + c^2(x)}, \]

with lower bound attained for Gaussians \( x_*(t) = e^{\alpha t^2 + \beta t + \gamma} \), \( \text{Re}\{\alpha\} < 0 \).

1. **Terminology**: waveforms \( x_* \) attaining the lower bound correspond to “squeezed states” in quantum mechanics and “linear chirps” in signal theory

2. **Interpretation**: possibility of localization in the plane beyond pointwise energy concentration
“chirp” localization
1. **Effective localization**: in the Wigner case, we have

\[
\lim_{Re\{\alpha\} \to 0} W_{x^*}(t, \omega) = \delta(\omega - (\beta + 2 \text{Im}\{\alpha\} t))
\]

2. **Generalization**: localization on more “arbitrary” curves of the plane by modifying the symmetry rules underlying the Wigner distribution (Gonçalvès and F., ’96)

3. **Caveat**: global localization holds for monocomponent signals only

4. **Way out**: reassignment (Kodera et al., ’76)
Starting with the smoothing relationship

\[ S_x^h(t, \omega) = \int \int W_x(s, \xi) W_h(s - t, \xi - \omega) \, dt \, \frac{d\omega}{2\pi}, \]

the key idea is

1. to replace the geometrical center of the smoothing time-frequency domain by the center of mass of the Wigner distribution over this domain, and

2. to reassign the value of the smoothed distribution to this local centroïd:

\[ \hat{S}_x^h(t, \omega) = \int \int S_x^h(\tau, \xi) \delta \left( t - \hat{t}_x(\tau, \xi), \omega - \hat{\omega}_x(\tau, \xi) \right) \, d\tau \, \frac{d\xi}{2\pi} \]
From uncertainty... to localization
Spectrogram geometry
sharp localization reassignment chirps, Hermite and logons

**spectrogram = smoothed Wigner**
From uncertainty... to localization: Spectrogram geometry.

sharp localization reassignment chirps, Hermite and logons

spreading of auto-terms

Wigner-Ville

spectrogram
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sharp localization reassignment chirps, Hermite and logons

cancelling of cross-terms

Wigner-Ville

spectrogram

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reassignment (Kodera et al., ’76, Auger & F., ’95)

Wigner-Ville

reassigned spectrogram
reassignment in action

1. **Spectrogram**: implicit computation of the local centroïds (Auger & F., ’95) via two additional short-time Fourier transforms (STFTs) with the companion windows $(T h)(t) = t h(t)$ and $(D h)(t) = (dh/dt)(t)$

2. **Gaussian spectrogram**: alternative computation of the local centroïds via partial derivatives of the magnitude of the STFT (Auger, Chassande-Mottin & F., ’12)

3. **Beyond spectrograms**: possible generalizations to other smoothings (smoothed pseudo-Wigner-Ville, scalogram, etc.)
**linear chirps**

- **Framework**: Gaussian window \( h(t) = \pi^{-1/4} e^{-t^2/2} \)
- **Result**: the linear chirp parameterized by
  \[
  \alpha = -\frac{1}{2} \left( \frac{1}{T^2} - ia \right) ; \quad \beta = 0
  \]

  is such that
  \[
  \lim_{T \to \infty} \hat{S}_h^x(t, \omega) = \frac{1}{2\pi} \delta(\omega - at)
  \]

- **Interpretation**: perfect localization
Hermite functions

- **Framework**: Gaussian window $g(t) = \pi^{-1/4} e^{-t^2/2}$

- **Hermite functions**:

  $$h_k(t) = C_k^{-1} \frac{1}{\sqrt{T}} H_{k-1}(\sqrt{2\pi} t / T) e^{-\pi(t/T)^2},$$

  with

  $$H_n(\alpha) = (-1)^n e^{\alpha^2} \frac{d^n}{d\alpha^n} e^{-\alpha^2}, \quad n = 0, 1$$

- **Radial reassignment**:

  $$\hat{\omega}(t, \omega) = \omega, \quad \frac{\hat{t}(t, \omega)}{t} = \frac{\omega}{t}$$
Hermite functions

- **Result** (F., *JFFA* ’12):

  \[ \hat{S}_{h_k}^{(g)}(t, \omega) = \frac{1}{2^{k-1}(k-1)!} \hat{V} \left( \sqrt{t^2 + \omega^2} \right) 1_D(t, \omega), \]

  with \( \hat{V} \neq \delta \) and:

  \[ D = \{(t, \omega) \mid t^2 + \omega^2 \geq t_m^2 + \omega_m^2 = 2(k - 1)\} \]

- **Interpretation**: clearance area and no perfect localization

- **Asymptotic localization**:

  \[ \hat{S}_{h_k}^{(g)}(t, \omega) \xrightarrow{k \to \infty} \delta(t^2 + \omega^2 - 2(k - 1)) \]
Hermite functions

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sharp localization reassignment chirps, Hermite and logons

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Uncertainty and Spectrogram Geometry
Hermite functions

order = 2 (theory)

order = 2 (simulation)
Hermite functions

M = 7 – Wigner

spectro.

reass. spectro.
From uncertainty... to localization
Spectrogram geometry
sharp localization reassignment chirps, Hermite and logons

Hermite functions

order = 7 (theory)

order = 7 (simulation)
Hermite functions

M = 18 – Wigner

spectro.

reass. spectro.
Hermite functions

order = 18 (theory)

order = 18 (simulation)
logons

- **Framework**: Gaussian window \( h(t) = \pi^{-1/4} e^{-t^2/2} \)
- **Result**: the Gabor “logon”, i.e., the previous linear chirp with \( a = 0 \), is such that

\[
S_x^h(t, \omega) = e^{-\frac{1}{2}(t^2+\omega^2)} \Rightarrow \Delta_{t\omega}(S_x^h) = 2
\]

\[
W_x(t, \omega) = 2 e^{-(t^2+\omega^2)} \Rightarrow \Delta_{t\omega}(W_x) = 1
\]

\[
\hat{S}_x^h(t, \omega) = 4 e^{-2(t^2+\omega^2)} \Rightarrow \Delta_{t\omega}(\hat{S}_x^h) = \frac{1}{2}
\]

- **Interpretation**: Heisenberg defeated?
resolving the paradox

1. **General remark**: as for Fourier, sharp localization not to be confused with resolution (i.e., ability to separate closely spaced components)

2. **Complete picture**: reassignment = squeezed distribution + vector field \( r_x(t, \omega) = (\hat{t}_x(t, \omega) - t, \hat{\omega}_x(t, \omega) - \omega)^t \) such that

   \[
   r_x(t, \omega) = \frac{1}{2} \nabla \log S^h_x(t, \omega)
   \]

3. **Interpretation**: basins of attraction

4. **Open question**: geometry of such basins and relationship with Heisenberg?
Bargmann representation: with a "circular" Gaussian window, the STFT can be expressed as

\[ F_h^h(z) = F^h_x(z) e^{-|z|^2/4}, \]

where \( z = \omega + j t \) and \( F^h_x(z) \) is an entire function of order at most 2.

Weierstrass-Hadamard factorization: it follows that

\[ F^h_x(z) = e^{Q(z)} \prod_n (1 - \tilde{z}_n) \exp \left( \tilde{z}_n + \tilde{z}_n^2/2 \right), \]

where \( Q(z) \) is a quadratic polynomial and \( \tilde{z}_n = z/z_n \).

Interpretation: complete characterization by zeroes and, by duality, by local maxima.
Result (Balasz et al., ’11)

“Universal” singularity of the phase derivative around zeroes of the STFT
In the Gaussian case, simple explanation based on:

**Proposition (Auger et al., '12)**

*Phase and magnitude of a Gaussian STFT form a Cauchy pair, i.e.,*

\[
\frac{\partial \Phi^h_x}{\partial t}(t, \omega) = \frac{\partial}{\partial \omega} \log M^h_x(t, \omega)
\]

\[
\frac{\partial \Phi^h_x}{\partial \omega}(t, \omega) = -\frac{\partial}{\partial t} \log M^h_x(t, \omega),
\]

*with* \( F^h_x(t, \omega) = M^h_x(t, \omega) e^{i\Phi^h_x(t, \omega)} e^{-(t^2+\omega^2)/4} \)
phase singularity

1. **Magnitude**: local behaviour in the vicinity of a zero \((t_0, \omega_0)\)

\[
M^h_x(t, \omega) \propto \left[ (\omega - \omega_0)^2 + (t - t_0)^2 \right]^{1/2}
\]

2. **Phase from magnitude**: “universal” hyperbolic divergence of the phase derivative

\[
\left. \frac{\partial \Phi^h_x}{\partial t}(t_0, \omega) \right|_{\omega \sim \omega_0} \sim (\omega - \omega_0)^{-1}
\]

\[
\left. \frac{\partial \Phi^h_x}{\partial \omega}(t, \omega_0) \right|_{t \sim t_0} \sim (t_0 - t)^{-1}
\]
zeroes $\leftrightarrow$ “phase dislocations” (Nye & Berry, ’74)
phase dislocation (time domain)
From uncertainty... to localization Spectrogram geometry on extrema Voronoi and Delaunay a simplified model

phase dislocation (complex plane)
phase dislocation (zoom)
Proposed approach

1. **Data**: white Gaussian noise
2. **Time-frequency representation**: Gaussian spectrogram
3. **Characterization**: identification of local extrema (zeroes and maxima) + Voronoi tessellation and Delaunay triangulation
4. **Analysis**: distribution of cell areas, lengths between extrema and heights of local maxima
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an example

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mean arrangement

1. **Observation**: average connectivity $\approx 6 \Rightarrow$ tiling with hexagonal cells
2. **Interpretation**: maximum packing of circular patches
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predictions

\[ D_M/d_m = \sqrt{3}; \quad N_M/N_m = 1/3; \quad A_M/A_m = 3 \]
From uncertainty... to localization

Spectrogram geometry on extrema Voronoi and Delaunay a simplified model

Simulation results

<table>
<thead>
<tr>
<th>Maxima</th>
<th>Distance: 5.24</th>
<th>Number: 24</th>
<th>Area: 17.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minima</td>
<td>Distance: 3.17</td>
<td>Number: 70</td>
<td>Area: 6.3</td>
</tr>
</tbody>
</table>

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comparison with model

\[
\frac{(d_M/d_m)}{\sqrt{3}}
\]

\[
\frac{(A_m/A_M)^3}{3}
\]

\[
\frac{(A_M/d_m^2)}{(\sqrt{3}/2)}
\]

C/6

mode theory median mean

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1 **Distributions**: significant dispersion for lengths and areas

2 **Ratios max/min**: reasonable agreement between experimental results and theoretical predictions

3 **Ranges of values**: if we call “effective domain” of the minimum uncertainty logon the circular domain which encompasses 95% of its energy, its radius and area are equal to $\sim 2.6$ and $21.8$, to be compared to the values $d_M/\sqrt{3} \sim 3$ and $2\pi/(3\sqrt{3})A_M \sim 21.8$ attached to the hexagonal tiling

4 **Interpretation**: tiling cells $\sim$ minimum uncertainty logons
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local maxima and Voronoi cells areas (1)

1. **Heights**: well-described by a Gamma distribution
2. **Areas**: idem and similar to the heights distribution for a proper renormalization

\[
2 \leq \iint_A (t^2 + \omega^2) |F_x^h(t, \omega)|^2 dt \frac{d\omega}{2\pi} \leq \left( \frac{|F|_* R^2(A)}{2} \right)^2
\]

**Proposition**

*The value \(|F|_*\) of a local maximum of the STFT magnitude and the area \(A\) of the associated Voronoi cell satisfy the uncertainty-type inequality*

\[
A.|F|_* \geq 3\sqrt{6}
\]
From uncertainty... to localization

Spectrogram geometry on extrema, Voronoi and Delaunay, a simplified model

local maxima and Voronoi cells areas (2)

- $\times \times \times$: data
- $\pm \pm \pm$: Gamma fit

Joint p.d.f.

areas of Voronoi cells (red) and local maxima of STFT magnitude (blue)

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Towards a random Gabor expansion

Previous results suggest a possible modeling of Gabor spectrograms as

\[ S^h_x(t, \omega) = \left| \sum_m \sum_n c_{mn} F^h(t - t_m, \omega - \omega_n) \right|^2 \]

with

1. locations \((t_m, \omega_n)\) distributed on some suitable randomized version of a triangular grid
2. magnitudes of the weights \(c_{mn}\) Gamma-distributed
3. locations and weights partly correlated
concluding remarks

1. Spectrogram: geometry constrained by time-frequency uncertainty
2. Reassignment: access to simplified properties, both geometrical and statistical
3. Gabor case: complete characterization by zeroes or local maxima?
4. Modeling: new ways of sparsifying energy distributions?