Means, Covariance, Fisher Information: the quantum theory and uncertainty relations

Paolo Gibilisco

Department of Economics and Finance
University of Rome “Tor Vergata”
gibilisco@volterra.uniroma2.it

November 21, 2012
History and credits

What follows originated from some works by

- S. Luo, Quantum Fisher information and uncertainty relations, *LMP*, 2000.

Other people involved:

Z. Zhang, Q. Zhang, Kosaki, Yanagi, Furuichi, Kuriyama, Gibilisco, Imparato, Isola, Hansen, Andai, Petz, Hiai, Szabo, Audenaart, Cai.
Heisenberg uncertainty principle

\[ A, B \in \mathcal{M}_{n,sa}(\mathbb{C}), \quad \rho \text{ density matrix} \]

\[ [A, B] := AB - BA \quad \mathbb{E}_\rho(A) := \text{Tr}(\rho A) \]

\[ \text{Var}_\rho(A) := \mathbb{E}_\rho(A^2) - \mathbb{E}_\rho(A)^2 \]

Heisenberg uncertainty principle (1927) reads as

\[ \text{Var}_\rho(A) \cdot \text{Var}_\rho(B) \geq \frac{1}{4} |\text{Tr}(\rho [A, B])|^2. \]
In classical probability, let \((X, Y)\) be a r.v. on \((\Omega, \mathcal{G}, p)\). The covariance matrix of \((X, Y)\) is symmetric and semidefinite positive so its determinant is non-negative and therefore

\[
\text{Var}_p(X) \cdot \text{Var}_p(Y) \geq \text{Cov}_p(X, Y)^2.
\]

So to have a general bound for \(\text{Var}_p(X) \cdot \text{Var}_p(Y)\) does not seems such a "quantum" phenomenon.
Quantum covariance

Quantum covariance (Schrödinger & Robertson)

\[
\text{Cov}_\rho(A, B) := \frac{1}{2} \text{Tr}(\rho(AB + BA)) - \text{Tr}(\rho A) \cdot \text{Tr}(\rho B) = \\
\text{Tr} \left[ \left( \frac{L_\rho + R_\rho}{2} \right) (A_0)B_0 \right].
\]

where \( A_0 := A - \text{Tr}(\rho A) \cdot I \) and

\[
L_\rho(A) := \rho A \quad R_\rho := A\rho
\]
Schrödinger and Robertson (1929-1930) improved UP

\[ \text{Var}_\rho(A) \cdot \text{Var}_\rho(B) \geq \text{Cov}_\rho(A, B)^2 + \frac{1}{4} |\text{Tr}(\rho[A, B])|^2. \]

Namely

\[
\begin{vmatrix}
\text{Var}_\rho(A) & \text{Cov}_\rho(A, B) \\
\text{Cov}_\rho(B, A) & \text{Var}_\rho(B)
\end{vmatrix}
\geq
\begin{vmatrix}
-i\frac{1}{2} \text{Tr}(\rho[A, A]) & -i\frac{1}{2} \text{Tr}(\rho[A, B]) \\
-i\frac{1}{2} \text{Tr}(\rho[B, A]) & -i\frac{1}{2} \text{Tr}(\rho[B, B])
\end{vmatrix}
\]
Let $A_1, \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det \{\text{Cov}_\rho(A_h, A_j)\} \geq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\},$$

for $h, j = 1, \ldots, N$

$\det \{\text{Cov}_\rho(A_h, A_j)\}$ is the \textit{generalized variance} of the random vector $(A_1, \ldots, A_n)$.
Robertson general UP (2nd version)

The matrix \(-\frac{i}{2} \text{Tr}(\rho[A_h, A_j])\) is anti-symmetric. Therefore, the Robertson UP reads as

\[
\det \{\text{Cov}_\rho(A_h, A_j)\} \geq \begin{cases} 
0, & N \text{ odd} \\
\det\{-\frac{i}{2} \text{Tr}(\rho[A_h, A_j])\}, & N \text{ even},
\end{cases}
\]

**Remark** If \(N = 2m + 1\), UP says (classically !) that the generalized variance is non-negative.
First problem: searching an UP for $N$ odd

- Robertson UP is based on the commutator $[A_h, A_j]$. If $N = 1$ this structure becomes meaningless!
- Intuitively, an UP for $N$ odd should be based on a structure which involves $[\rho, A]$.
- This commutator appears in quantum dynamics.
(Elementary) Quantum dynamics

Let $\rho(t)$ be a curve in $\mathcal{D}_n^1$ and let $H \in M_{n,sa}$; $\rho(t)$ satisfies Schrödinger equation w.r.t. $H$ if

$$\dot{\rho}(t) = \frac{d}{dt} \rho(t) = i[\rho(t), H].$$

Equivalently, $\rho_H(t)$, the time evolution of $\rho = \rho_H(0)$ determined by $H$, evolves according to the formula

$$\rho_H(t) := e^{-itH} \rho e^{itH}.$$

Therefore

$$\dot{\rho}_H(0) = i[\rho, H]$$
Second problem: different quantum covariances?

\[ \text{Cov}_\rho(A, B) := \frac{1}{2} \text{Tr}(\rho(AB + BA)) - \text{Tr}(\rho A) \cdot \text{Tr}(\rho B) = \]
\[ = \text{Tr} \left[ \left( \frac{L_\rho + R_\rho}{2} \right)(A_0)B_0 \right]. \]

Is the above definition “natural”?
Certainly it coincides with the classical covariance in a commutative setting.
It uses the ”arithmetic mean” of the left and right multiplication operator

\[ m_{\text{arith}}(L_\rho, R_\rho) := \frac{L_\rho + R_\rho}{2} \]

This suggests that we may consider other noncommutative ”‘means”.

P. Gibilisco (Rome “Tor Vergata”) 
Means, Covariance, Fisher Information 
November 2012
Harmonic covariance?

If we consider the “harmonic” covariance

$$\text{Cov}_{\rho}^{\text{har}}(A, B) := \text{Tr} \left((2(L_{\rho}^{-1} + R_{\rho}^{-1})^{-1}) (A_0)B_0\right),$$

also this coincides with the classical definition where there is no difference between $L_{\rho}$ and $R_{\rho}$!

Is there a quantum criterion to prefer a certain covariance (mean)?
Means for numbers ...

Let $\mathbb{R}^+ = (0, +\infty)$.

A *mean* for pair of positive numbers is a function $m(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

i) $m(x, x) = x$;
ii) $m(x, y) = m(y, x)$;
iii) $x \leq x', y \leq y' \implies m(x, y) \leq m(x', y')$;
iv) for $t > 0$ one has $m(tx, ty) = t \cdot m(x, y)$;
v) $m(\cdot, \cdot)$ is continuous.

$$\mathcal{M}_{nu} := \{ m(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ | m \text{ is a mean} \}$$
... from certain functions

$\mathcal{F}_{nu}$ is the class of functions $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

iii) $f(1) = 1$;
iv) $tf(t^{-1}) = f(t)$;

iii) $x \leq x' \implies f(x) \leq f(x')$;
iv) $f$ is continuous.

**Proposition**

There is bijection between $\mathcal{M}_{nu}$ and $\mathcal{F}_{nu}$ given by the formula

$$m_f(x, y) := yf(xy^{-1})$$
Operator means

Kubo-Ando 1980
Let $D_n := \{ A \in M_n | A > 0 \}$.

A mean is a function $m : D_n \times D_n \rightarrow D_n$ such that

(i) $m(A, A) = A$,
(ii) $m(A, B) = m(B, A)$,
(iii) $A < A', B < B' \implies m(A, B) < m(A', B')$,
(iv) $m$ is continuous,
(v) $Cm(A, B)C^* \leq m(CAC^*, CBC^*)$, for every $C \in M_n$.

Property (vi) is the transformer inequality.
Operator monotone functions

\( M_n = \text{complex matrices} \)

**Definition**

\( f : (0, +\infty) \to R \) is operator monotone iff \( \forall A, B \in M_n \) and \( \forall n = 1, 2, ... \)

\[
0 \leq A \leq B \quad \implies \quad 0 \leq f(A) \leq f(B).
\]

**Definition**

\( \varphi \) is a Pick function if it is analytic in the upper half plane and map the latter into itself.
Löwner Theorem

Löwner 1932

Theorem

$f$ is operator monotone iff it is the restriction of a Pick function.
Usually one consider o.m. functions that are:
i) normalized i.e. \( f(1) = 1 \);
ii) symmetric i.e. \( tf(t^{-1}) = f(t) \).

\( \mathcal{F}_{op} \) := family of normalized symmetric o.m. functions.

**Examples**

\[
\begin{align*}
\frac{1 + x}{2}, & \quad \sqrt{x}, & \quad \frac{2x}{1 + x}.
\end{align*}
\]
Kubo–Ando theorem

\( \mathcal{M}_{op} := \text{family of matrix means}. \)

Kubo and Ando (1980) proved the following, fundamental result.

**Theorem**

There exists a bijection between \( \mathcal{M}_{op} \) and \( \mathcal{F}_{op} \) given by the formula

\[
m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}) A^{\frac{1}{2}}.
\]

\([A, B] = 0 \implies m_f(A, B) := Af(BA^{-1}).\)
Kubo–Ando inequality

Examples of operator means

\[
\frac{A + B}{2} \quad A^\frac{1}{2} \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^\frac{1}{2} \\
2(A^{-1} + B^{-1})^{-1}
\]

Fundamental inequality

\[
2(A^{-1} + B^{-1})^{-1} \leq m_f(A, B) \leq \frac{A + B}{2} \quad \forall f \in \mathcal{F}_{op}
\]
$g$-Covariance

To each operator monotone $g \in \mathcal{F}_{op}$ one associate the means $m_g(\cdot, \cdot)$.

Define the $g$-covariance as

$$Cov^g_\rho(A, B) := \text{Tr}(m_g(L_\rho, R_\rho)(A_0)B_0)$$

If

$$g(x) = \frac{1 + x}{2}$$

then

$m_g = \text{arithmetic mean}$

and $Cov^g_\rho(A, B)$ is the standard covariance introduced by Schrödinger and Roberston.
Fisher information for densities

\( X: \Omega \rightarrow \mathbb{R} \) real r. v. with diff. density \( \rho \)

The score is

\[ J_\rho := \frac{\rho'}{\rho} \]

The Fisher information is

\[ I_X := I_\rho = \text{Var}_\rho(J_\rho) = \int_{\mathbb{R}} \frac{(\rho')^2}{\rho} \]
$M$ statistical model (set of densities)

$M$ can be considered as a manifold where the $\rho'$s play the role of tangent vectors.

$I_\rho$ is associated to a Riemannian metrics as in the formula

$$g_{\rho,F}(\rho', \rho') = \int_{\mathbb{R}} \frac{\rho' \cdot \rho'}{\rho} = I_\rho$$
FI on the simplex

\[ \mathcal{P}_n^1 := \{ \rho \in \mathbb{R}^n | \sum_i \rho_i = 1, \ \rho_i > 0 \} \].

\[ T\mathcal{P}_n^1 = \{ u \in \mathbb{R}^n | \sum_i u_i = 0 \} \].

\[ g_{\rho, F}(u, v) := \sum_i \frac{u_i v_i}{\rho_i} \]

This will be the Fisher-Rao metric
Geodesic distance (Bhattacharya): \[ d_F(\rho, \sigma) = 2 \arccos \left( \sum_i \rho_i^{\frac{1}{2}} \sigma_i^{\frac{1}{2}} \right) \]
The link with entropy

i) Hessian of Kullback-Leibler relative entropy

\[ S(\rho, \sigma) := \sum_i \rho_i (\log \rho_i - \log \sigma_i); \]

\[- \frac{\partial^2}{\partial t \partial s} S(\rho + tu, \rho + sv) \bigg|_{t=s=0} = \sum_{i=1}^{n} \frac{u_i v_i}{\rho_i + sv_i} \bigg|_{t=s=0} = \sum_{i=1}^{n} \frac{u_i v_i}{\rho_i} = g_{\rho,F}(u, v). \]
The link with the sphere

FI as a spherical geometry (Rao, Dawid)

ii) pull-back of the map

\[ \varphi(\rho) = \varphi(\rho_1, \ldots, \rho_n) = 2(\sqrt{\rho_1}, \ldots, \sqrt{\rho_n}) \]

\[ g^\varphi_{\rho}(u, v) = g_{\varphi(\rho)}(D_\rho \varphi(u), D_\rho \varphi(v)) \]
\[ = \langle M_{\rho^{-1/2}}(u), M_{\rho^{-1/2}}(v) \rangle \]
\[ = \sum_{i=1}^{n} \frac{u_i v_i}{\rho_i} = g_{\rho,F}(u, v). \] (1)

This explains the geodesic distance (Bhattacharya):

\[ d_F(\rho, \sigma) = 2 \arccos \left( \sum \rho_i^{1/2} \sigma_i^{1/2} \right) \]
Look at Fisher information in different ways:

i) Hessian of Kullback-Leibler relative entropy

\[ K(\rho, \sigma) := \sum_i \rho_i (\log \rho_i - \log \sigma_i); \]

ii) pull-back of the map \( \rho \mapsto \sqrt{\rho} \);

iii) get the scores using the (Symmetric) Logarithmic Derivative

\[ \frac{\partial \rho(\theta)}{\partial \theta} = \frac{1}{2} \left( \frac{\partial}{\partial \theta} \log(\rho(\theta)) \cdot \rho(\theta) + \rho(\theta) \cdot \frac{\partial}{\partial \theta} \log(\rho(\theta)) \right) \]
... to a zoo of examples of QFI

Examples of quantum Fisher informations

Hessian of Umegaki relative entropy
\[ \text{Tr} (\rho (\log \rho - \log \sigma)) \]
\[ \rightarrow \text{BKM metric} \]

Pull-back of the map \( \rho \rightarrow \sqrt{\rho} \)
\[ \rightarrow \text{WY metric} \]

Symmetric logarithmic derivative
\[ \rightarrow \text{Bures-Uhlmann metric (SLD)} \]

Can we have a unified quantum approach?
Chentsov Theorem

Yes! Using the classical Chentsov theorem.

On the simplex $\mathcal{P}_n^1$ the Fisher information is the only Riemannian metric contracting under an arbitrary coarse graining $T$, namely for any tangent vector $X$ at the point $\rho$ we have

$$g^m_{T(\rho)}(TX, TX) \leq g^n_{\rho}(X, X)$$

**Remark**

Coarse graining = stochastic map = linear, positive, trace preserving.
Monotone metrics (or QFI) according Chensov-Morozova

\[ D^1_n := \{ \rho \in M_n | \text{Tr}(\rho) = 1, \rho > 0 \} = \text{faithful states} \]

Definition
A quantum Fisher information is a Riemannian metric on \( D^1_n \) contracting under an arbitrary coarse graining \( T \), namely

\[ g^m_T(\rho)(TA, TA) \leq g^n_\rho(A, A). \]

(quantum) coarse graining = linear, (completely) positive, trace preserving map.
Petz theorem

\[ L_\rho(A) := \rho A \quad R_\rho(A) := A\rho \]

Petz theorem

There is bijection among quantum Fisher information and operator monotone functions (and/or operator means) given by the formula

\[ \langle A, B \rangle_{\rho,f} := \text{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)). \]
Geodesic distances

Closed form of geodesics distances are known only in two cases

\[ D_{\text{Bures}}(\rho, \sigma) = 2 \arccos \text{Tr}(\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}})^{\frac{1}{2}} \]

\[ D_{\text{WY}} = 2 \arccos \text{Tr}(\rho^{\frac{1}{2}} \sigma^{\frac{1}{2}}) \]

(Gibilisco-Isola)
Kubo-Ando-Petz (Löwner)

\[ f \]

\[ m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}. \]

\[ \langle A, B \rangle_{\rho, f} := \text{Tr}(A \cdot m_f(L_{\rho}, R_{\rho})^{-1}(B)). \]
Decomposition of tangent space I

\[ T_\rho D_1^n = \{ A \in M_{n,sa} \mid A = A^*, \quad \text{Tr}(A) = 0 \}. \]

\[ T_\rho D_1^n = (T_\rho D_1^n)_c \oplus (T_\rho D_1^n)_o \]

where

\[ (T_\rho D_1^n)_c := \{ A \in T_\rho D_1^n \mid [\rho, A] = 0 \} \]

\[ (T_\rho D_1^n)_o := \text{orth. compl. of } (T_\rho D_1^n)_c \text{ resp. to H-S} \]
Decomposition of tangent space II

For each QFI and for each $A \in (T^\rho D_n^1)^c$ one has

$$\langle A, A \rangle_{\rho, f} = \text{Tr}(\rho^{-1} A^2).$$

To evaluate a QFI one has just to know what happens for $(T^\rho D_n^1)^o$ whose typical element has the form

$$i[\rho, A] \quad \text{A s.a.}$$
Regular and non-regular QFI

\[ \mathcal{F}_{op} := \{ f \text{ op. mon.} | f(1) = 1, \quad tf(t^{-1}) = f(t) \} \]

\[ \mathcal{F}_{op}^r := \{ f \in \mathcal{F}_{op} | f(0) := \lim_{t \to 0} f(t) > 0 \} \]

\[ \mathcal{F}_{op}^n := \{ f \in \mathcal{F}_{op} | f(0) = 0 \} \]

\[ \mathcal{F}_{op} = \mathcal{F}_{op}^r \cup \mathcal{F}_{op}^n \]

Remark The word non-regular should lead to a negative attitude: the BKM metric is non-regular but widely used in quantum statistical mechanics.

Why is this decomposition relevant?
Riemannian metrics on the sphere

\[ B_3 := \{ (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1 \} \]
\[ S_2 := \overline{B_3} \quad 0 := (0, 0, 0) \]
\[ \mathcal{M} := B_3/(S_2 \cup \{0\}) \]

\( \mathcal{M} \) is a fiber bundle over \( S_2 \) with projection

\[ \pi : \mathcal{M} \to S_2 \]
\[ \pi(x, y, z) := \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x, y, z) \]
RIEM. METRICS ON THE SPHERE II

\[ M \ni D_n \to \rho \in S_2 \text{ radially iff } \]
\[ \pi(D_n) = \rho \quad \forall n \text{ and } \lim D_n = \rho \]
Differential
\[ T\pi : T M \to TS_2 \]
Horizontal-Vertical decomposition
\[ T_{D}M = \text{Ker}(T_{D}\pi) \oplus H_D \]
\[ H_D = \text{horizontal tangent vectors at the point } D \]
Restriction

\[ T_D \pi = H_D \rightarrow T_\rho S_2 \]

is a linear isomorphism between \( H_D \) and \( T_\rho S_2 \) (where \( \rho = T(D) \)).

We may “lift” tangent vectors \( u, v \in T_\rho S_2 \) to \( u_D, v_D \in T_D \mathcal{M} \).
Suppose we have:
i) a Riemannian metric $g(\cdot, \cdot)$ on $\mathcal{M}$;
ii) a Riemannian metric $h(\cdot, \cdot)$ on $S_2$.

$h$ is the radial extension of $g$ if

\[ D_n \to \rho \quad \text{radially} \]

\[ g(u_{D_n}, v_{D_n}) \to k(u, v) \]
The Bloch sphere

$2 \times 2$ matrices, $I$ identity, $\sigma_1, \sigma_2, \sigma_3$ Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

Stokes parametrization of qubits

$$
\rho = \frac{1}{2} \left( I + \langle (x, y, z), (\sigma_1, \sigma_2, \sigma_3) \rangle \right)
$$

$$
x^2 + y^2 + z^2 \leq 1
$$
Petz-Sudar theorem

Pure states $\rightarrow x^2 + y^2 + z^2 = 1$ (the sphere $S_2$)

Faithful mixed states $\rightarrow x^2 + y^2 + z^2 < 1$

(manifold $\mathcal{M}$ plus the origin)

**Theorem**

If $\langle \cdot, \cdot \rangle_{FS}$ denotes the standard Riemannian metric on the sphere $S_2$ (pure states), then a QFI $\langle \cdot, \cdot \rangle_{\rho,f}$ has a radial extension iff it is regular. The extension is given by

$$\frac{1}{2f(0)} \langle \cdot, \cdot \rangle_{FS}$$
Remark

True in general using the *Fubini–Study* metric on the projective space $CP^n$.

More delicate because for $n > 2$:

extreme boundary (pure states) $\neq$ topological boundary ($\det \rho = 0$).
The function $\tilde{f}$

$$\tilde{f}(x) := \frac{1}{2} \left[ (x + 1) - (x - 1)^2 \frac{f(0)}{f(x)} \right]$$

**Theorem**

$f \in \mathcal{F}^r_{op}$ ($f$ is a regular n. s. o. m. function)

⇓

$\tilde{f} \in \mathcal{F}^n_{op}$ ($\tilde{f}$ is a non-regular n. s. o. m. function)

Moreover $f \rightarrow \tilde{f}$ is bijection.

Gibilisco-Imparato-Isola-Hansen
Regular and non-regular means

\[ f \rightarrow \tilde{f} \]

\[ m_f \rightarrow m_{\tilde{f}} \]

Examples

\[ \frac{x + y}{2} \rightarrow \frac{2}{\frac{1}{x} + \frac{1}{y}} \]

\[ \left( \frac{\sqrt{x} + \sqrt{y}}{2} \right)^2 \rightarrow \sqrt{xy} \]
$x > 0, \beta \in (0, \frac{1}{2})$

\[
f_\beta(x) := \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)}
\]

\[
\tilde{f}_\beta(x) = \frac{x^\beta + x^{1-\beta}}{2}
\]

Fix $x > 0$. Then $\tilde{f}_\beta(x)$ is decreasing as a function of $\beta$.
This remark allows a great simplification of a preceding result by Kosaki.
Fundamental formula

**Theorem**

If $f$ is regular then

$$
\frac{f(0)}{2} \langle i[\rho, A], i[\rho, B] \rangle_{\rho, f} = \text{Cov}_\rho(A, B) - \text{Cov}_{\tilde{f}}(A, B).
$$

An immediate consequence is

**Proposition**

$$
\text{Var}_\rho(A) \geq \frac{f(0)}{2} \| i[\rho, A] \|_{\rho, f}^2
$$

The above is the case $N = 1$ of the following ...
The dynamical UP

**Theorem**

Let $A_1 \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det \{ \text{Cov}_\rho(A_h, A_j) \} \geq \det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\}$$

for $h, j = 1, \ldots, N$, for all $f \in \mathcal{F}_{op}$.

Nontrivial bound also if $N$ is odd!
Interlude: the case $N = 1$ and WYD-information

Wigner-Yanase-Dyson information

A s.a. matrix (observable in QM)
$ho$ density matrix (state in QM)

$$I_{\rho}^\beta(A) := -\frac{1}{2} \text{Tr}([\rho^\beta, A][\rho^{1-\beta}, A]) \quad \beta \in (0, \frac{1}{2}]$$
Interlude: WYD applications

- strong subadditivity of entropy (Lieb-Ruskai, 1973)
- homogeneity of the state space of factors of type $\text{III}_1$ (Connes-Stormer, 1978);
- measures for quantum entanglement (Chen, 2005; Klyachko-Oztop-Shumovsky, 2006);
- uncertainty relations;
- quantum hypothesis testing (Calsamiglia et al., 2008)

(Explanation: WYD is a QFI)
Interlude: Lieb convexity

Theorem

\( \rho^\beta(A) \) is convex as function of \( \rho \).

“If two ensembles are united, the information content of the resulting ensemble should be smaller than the average information content of the component ensembles” (Wigner-Yanase original paper)
Remark

Since

\[ I_\rho^\beta(A) = \frac{1}{2} \text{Tr}(\rho A^2) - \text{Tr}(\rho^\beta A \rho^{1-\beta} A) \]

convexity of \( I_\rho^\beta(A) \) is equivalent to concavity of

\[ \rho \rightarrow \text{Tr}(\rho^\beta A \rho^{1-\beta} A) \]
Lieb convexity and SSA

Applications: Strong subadditivity of von Neumann quantum entropy

$$S(\rho) := -\text{Tr}(\rho \log \rho)$$

(Lieb & Ruskai 1975)

SSA implies strong results about to thermodynamic limit of entropy per unit volume (Robinson, Ruelle, & Lanford 1967-8)
Interlude: WYD as quantum Fisher information

Definition
The *metric adjusted skew information* is

\[ I^f_\rho (A) := \frac{f(0)}{2} ||i[\rho, A]||^2_{\rho,f} \]

Proposition
In the case \( x > 0, \beta \in (0, \frac{1}{2}] \)

\[ f_\beta(x) := \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)} \]

we have

\[ I^{f_\beta}_\rho (A) := \frac{f_\beta(0)}{2} ||i[\rho, A]||^2_{\rho,f_\beta} = -\frac{1}{2} \text{Tr}([\rho^\beta, A][\rho^{1-\beta}, A]) \]
Interlude: concavity-convexity trick

$T(\cdot), S(\cdot)$ real functions on states

$S(\cdot)$ concave, $T(\cdot)$ convex

$S(\cdot) = T(\cdot)$ on pure states

$\Downarrow$

$S(\rho) \geq T(\rho) \quad \forall \rho \in \mathcal{D}_n^1$
Interlude: $N = 1$, the DUP from Lieb convexity

Theorem (F. Hansen 2006)

$$\text{Var}_\rho(A) \geq I^f_\rho(A)$$

Proof

$\text{Var}_\rho(A)$ is concave

$I^f_\rho(A)$ is convex

$\rho$ pure implies $\text{Var}_\rho(A) = I^f_\rho(A)$

___

Warning: this elegant proof do not work in the general case of the DUP!
Why “dynamical”? 

Let $\rho > 0$ be a state and $H, K \in M_{n,sa}$. Suppose that $\rho = \rho_H(0) = \rho_K(0)$. Then, for any $f \in F_{op}$, one has (taking the square root of both sides of the DUP)

$$\text{Area}_{\rho}^{\text{Cov}}(H, K) \geq \frac{f(0)}{2} \cdot \text{Area}_{\rho}^{f}(\dot{\rho}_H(0), \dot{\rho}_K(0)).$$

The bound on the right side of the inequality can be seen as a measure of the difference between the dynamics generated by $H$ and $K$. 
How the bound in the DUP depends on $f$

**Theorem**

Define for $f \in F_{op}$

$$S(f) := \det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\}$$

$$\tilde{f}(x) := \frac{1}{2} \left[ (x + 1) - (x - 1)^2 \frac{f(0)}{f(x)} \right].$$

Then, for any $f, g \in F_{op}$

$$\tilde{f} \leq \tilde{g} \implies S(f) \geq S(g).$$
The optimal bound

Let $f_{SLD}(x) := \frac{1 + x}{2}$. Since for any $f \in \mathcal{F}_{op}^r$

\[
\frac{2x}{1 + x} = \tilde{f}_{SLD} \leq \tilde{f}
\]

then

$S(f_{SLD}) \geq S(f)$

namely the optimal bound is given by Bures-Uhlmann metric.
Relation with standard UP - 1

Let \( f \in \mathcal{F}_{op}^r \). The inequality

\[
\det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\} \geq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}
\]

is (in general) false for any \( N = 2m \).

Remark
The proof is a consequence of Hadamard inequality:

\[
\det(H) \leq \prod_{j=1}^{N} h_{jj}
\]

for any \( H \in M_{N,sa} \).
Let $f \in \mathcal{F}_{op}$. The inequality

$$\det \left\{ \frac{f(0)}{2} \langle i[\rho, A_h], i[\rho, A_j] \rangle_{\rho, f} \right\} \leq \det \left\{ -\frac{i}{2} \text{Tr}(\rho[A_h, A_j]) \right\}$$

is (in general) false for any $N = 2m$. 
The dynamical UP ($g$-version)

**Theorem**

Let $A_1, \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det \{ \text{Cov}^g_{\rho}(A_h, A_j) \} \geq \det \{ g(0)f(0)\langle [\rho, A_h], i[\rho, A_j] \rangle \}_{\rho, f}$$

for $h, j = 1, \ldots, N$,

for all $g, f \in \mathcal{F}_{op}$. 
Conclusion - Dynamical UP case

For the dynamical UP quantum $g$-covariances coming from regular $g$ (constant $g(0) \neq 0$) do have uncertainty relations.

Quantum $g$-covariances coming from nonregular $g$ (constant $g(0) = 0$) do NOT have (non-trivial) uncertainty relations.
Theorem

Let $A_1, \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det \left\{ \text{Cov}^g_{\rho}(A_h, A_j) \right\} \geq \det \left\{ -i \cdot g(0) \cdot \text{Tr}(\rho [A_h, A_j]) \right\},$$

for $h, j = 1, \ldots, N$,
for all $g \in \mathcal{F}_{op}$.

Remark: $g(0)$ is the best constant in the above inequalities.
Quantum $g$-covariances coming from regular $g$ (constant $g(0) \neq 0$) do have uncertainty relations.
Quantum $g$-covariances coming from nonregular $g$ (constant $g(0) = 0$) do NOT have uncertainty relations.
The usual quantum covariance has the most demanding one (since $g(0) = \frac{1}{2}$ only for the arithmetic mean).

After all Schrödinger and Robertson were right ...
Post scriptum: The DUP on von Neumann algebras

While the first papers on the DUP where forced to use eigenvalues (see the complicated calculations in the paper by Kosaki) now one has to generalize something like the mean of the operators $L_\rho$ and $R_\rho$. These are commuting operators therefore

$$m_{\tilde{f}}(L_\rho, R_\rho) = L_\rho \tilde{f}(R_\rho L_\rho^{-1}) = L_\rho \tilde{f}(\Delta_\rho)$$

So we are dealing with the modular operator and this construction makes sense in the general setting of von Neumann algebras.

Gibilisco-Isola

Petz-Szabo