Noncommutative solenoids and their projective modules

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November 20, 2012
Operator Algebras and Time-Frequency Methods Workshop
Erwin Schrödinger Institute, Vienna
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Joint Work with
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Outline

1. Introduction and Background
2. NCS as twisted group $C^*$-algebras and as direct limit algebras
3. The range of the trace and projective modules
Twisted group algebras and transformation group $C^*$-algebras have been studied since the early 1960’s. Much progress has been made in studying such $C^*$-algebras when the groups involved are finitely generated (or compactly generated, in the case of Lie groups). Even when $G = \mathbb{Z}^n$, these $C^*$-algebras give a rich class of examples. Some of the names involved in these studies include L. Baggett and A. Kleppner, M. Rieffel, G. Elliott and D. Evans, and S. Echterhoff and J. Rosenberg.

Here we study twisted group $C^*$-algebras corresponding to $N$-adic rational numbers, $\mathbb{Q}_N$. The dual of this group is the $N$-solenoid, so we call these $C^*$-algebras noncommutative solenoids!!
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Here we study twisted group $C^*$-algebras corresponding to $N$-adic rational numbers, $\mathbb{Q}_N$. The dual of this group is the $N$-solenoid, so we call these $C^*$-algebras noncommutative solenoids!!
Let \( N \) be a positive integer, with \( N > 1 \). Let \( \mathbb{Q}_N = \bigcup_{j=0}^{\infty} (\mathbb{N}^{-j}(\mathbb{Z})) \subset [\mathbb{Q}] \).

The Pontryagin dual of \( \mathbb{Q}_N \) is the compact inverse limit abelian group \( \{ (z_i)_{i=0}^{\infty} : z_i \in \mathbb{T}, (z_{i+1})^N = z_i, \forall i \} \). We call this group the \( N \)-solenoid, and denote it by \( S_N \).

The dual pairing is defined by
\[
\langle N^{-j}(k), (z_i)_{i=0}^{\infty} \rangle = (z_j)^k.
\]
For every \( j \in \mathbb{N} \cup \{0\} \), there is a map \( \pi_j : S_N \rightarrow \mathbb{T} \) given by \( \pi_j((z_i)_{i=0}^{\infty}) = z_j \).
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Notation and initial definitions:

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Multipliers on $\mathbb{Q}_N \times \mathbb{Q}_N$

Just as $\mathbb{Z}$ has no nontrivial multipliers, neither does $\mathbb{Q}_N$. However moving up in rank, taking $\Gamma = \mathbb{Q}_N \times \mathbb{Q}_N$, there are non-trivial multipliers. We thus study the corresponding twisted group $C^*$-algebras $C^*(\Gamma, \sigma)$, for $\sigma \in \mathbb{Z}^2(\Gamma, \mathbb{T})$, a multiplier on $\Gamma$ with values in the circle group $\mathbb{T}$. Following the model for the case where $\Gamma = \mathbb{Z}^2$, we call these $C^*$-algebras noncommutative solenoids. We calculate the non-trivial multipliers on $\Gamma$:

**Theorem**

$(L-P)$ Let $\Gamma = \mathbb{Q}_N \times \mathbb{Q}_N$ and let $\sigma$ be a multiplier on $\Gamma$. Then $\sigma$ is cohomologous to a multiplier of the form $\Psi_\alpha : \Gamma \times \Gamma \to \mathbb{T}$; for $\alpha = (\alpha_i) \in \Xi_N \subset \prod_{i=1}^\infty [0, 1)_i : \forall i \in \mathbb{N}, N\alpha_{i+1} = \alpha_i + k$ for some $k \in \{0, 1, \ldots, N - 1\}$, 

$$
\Psi_\alpha((\frac{p_1}{N^{k_1}}, \frac{p_2}{N^{k_2}}), (\frac{p_3}{N^{k_3}}, \frac{p_4}{N^{k_4}})) = e^{2\pi i(\alpha(k_1+k_4)p_1p_4)}.
$$
One can show that for $\alpha, \beta \in \Xi_N$, then the cohomology classes of $\alpha$ and $\beta$ are equal in $H^2(\Gamma, \mathbb{T})$ if and only if $\alpha_i = \beta_i$, $\forall i \in \mathbb{N}$. As a set, $\Xi_N$ can be identified with the $N$-solenoid $S_N$, but we find the additive version $\Xi_N$ and modular arithmetic easier to use in calculations that $S_N$. As a topological group, $H^2(\Gamma, \mathbb{T})$ can be identified with $S_N$.

The proof of this involves the standard calculations for cohomology classes of multipliers on discrete abelian groups, due to A. Kleppner, generalizing results of Backhouse and Bradley.
One can show that for \( \alpha, \beta \in \Xi_N \), then the cohomology classes of \( \alpha \) and \( \beta \) are equal in \( H^2(\Gamma, \mathbb{T}) \) if and only if \( \alpha_i = \beta_i, \ \forall i \in \mathbb{N} \). As a set, \( \Xi_N \) can be identified with the \( N \)-solenoid \( S_N \), but we find the additive version \( \Xi_N \) and modular arithmetic easier to use in calculations that \( S_N \). As a topological group, \( H^2(\Gamma, \mathbb{T}) \) can be identified with \( S_N \).

The proof of this involves the standard calculations for cohomology classes of multipliers on discrete abelian groups, due to A. Kleppner, generalizing results of Backhouse and Bradley.
Let $\Gamma = \mathbb{Q}_N \times \mathbb{Q}_N$, and let $\alpha \in \Xi_N$. Form the twisted group $C^*$-algebra $C^*(\Gamma, \psi_\alpha)$, where recall that $C^*(\Gamma, \psi_\alpha)$ is the $C^*$-completion of the involutive Banach algebra $l^1(\Gamma, \psi_\alpha)$, where the convolution is given by

$$f_1 * f_2(\gamma) = \sum_{\gamma_1 \in \Gamma} f_1(\gamma_1) f_2(\gamma - \gamma_1) \psi_\alpha(\gamma_1, \gamma - \gamma_1),$$

and

$$f^*(\gamma) = \psi_\alpha(\gamma, -\gamma) f(-\gamma).$$

By definition, the $C^*$-enveloping algebra of $l^1(\Gamma, \psi_\alpha)$ is the twisted group algebra $C^*(\Gamma, \psi_\alpha)$. 
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Necessary and sufficient conditions for simplicity of $C^*(\Gamma, \Psi_\alpha)$

Our $C^*$-algebras, originally viewed as twisted group algebras for discrete abelian, now have a representation as transformation group $C^*$-algebras. Either way, there are longstanding methods in place to determine whether or not these $C^*$ algebras are simple:

**Definition**

For a multiplier $\sigma : \Gamma \times \Gamma \to \mathbb{T}$ on the discrete abelian group $\Gamma$, the symmetrizer subgroup of $\Gamma$ corresponding to $\sigma$ is defined by

$$S_\sigma = \{ \gamma \in \Gamma : \sigma(\gamma, g)\sigma(g, \gamma)^{-1} = 1, \forall g \in \Gamma \}.$$ 

It is well known that $C^*(\Gamma, \sigma)$ is simple if and only if $S_\sigma$ is trivial.
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It is well known that $C^*(\Gamma, \sigma)$ is simple if and only if $S_\sigma$ is trivial.
Upon computing when $S_\sigma = \{0\}$, we obtain:

**Theorem**

(L.-P.) Let $\Gamma = \mathbb{Q}_N \times \mathbb{Q}_N$, let $\alpha \in \Xi_N$, and let $\psi_\alpha$ be the multiplier on $\Gamma$ defined previously. Then the following are equivalent:

1. $C^* (\Gamma, \psi_\alpha)$ is simple;
2. The set $\{\alpha_i : i \in \mathbb{N}\}$ is infinite;
3. For every $k \in \mathbb{N}$, $(N^k - 1)\alpha_0 \notin \mathbb{Z};$

The dual group of $\Gamma$, $S_N \times S_N$, has a natural dual action on $C^* (\Gamma, \psi_\alpha)$. So, there is a invariant trace on $C^* (\Gamma, \psi_\alpha)$ that is unique in the simple case.
Example: the aperiodic rational case

Note in the theorem above that it is possible for all the $\alpha_i$ to be rational, yet have $S_\alpha$ still be trivial!

Example

Let $N = 7$, and consider $\alpha \in \Xi_7$ given by

$$\alpha = \left( \frac{2}{7}, \frac{2}{49}, \frac{2}{343}, \frac{2}{2401}, \cdots, \frac{2}{7^n}, \cdots, \right),$$

Note that $\alpha_i \in \mathbb{Q}, \forall i$, yet the theorem tells us that $C^*(\Gamma, \Psi_\alpha)$ is simple!
Introduction and Background

NCS as twisted group C*-algebras and as direct limit algebras
The range of the trace and projective modules

Summary

Relationship to noncommutative tori

Example

Recall for $\theta \in [0, 1)$, the rotation algebra $A_\theta$ is the universal C*-algebra generated by unitaries $U, V$ satisfying

$$UV = e^{2\pi i \theta} VU.$$ 

The rotation algebra $A_\theta$ can be realized in a variety of ways, including as a twisted group C*-algebra $C^*(\mathbb{Z}^2, \sigma_\theta)$, for

$$\sigma_\theta((m_1, n_1), (m_2, n_2)) = e^{2\pi i n_1 m_2 \theta}.$$ 

$A_\theta$ is simple if and only if $\theta$ is irrational.

Recall that the $A_\theta$ are called “noncommutative tori".
Realizing $C^*(\Gamma, \psi_\alpha)$ as a direct limit of noncommutative tori

**Theorem**

(L.-P.) Let $N \in \mathbb{N}$ with $N > 1$ and $\alpha \in \Xi_N$. Let $A_\theta$ denote the rotation $C^*$-algebra for the rotation of angle $2i\pi \theta$. For all $n \in \mathbb{N}$, let $\varphi_n$ be the unique $*$-morphism from $A_{\alpha 2n}$ into $A_{\alpha 2n+2}$ given by:

\[ \begin{align*} U_{\alpha 2n} &\mapsto -U_{\alpha 2n}^N \\ V_{\alpha 2n} &\mapsto -V_{\alpha 2n}^N \end{align*} \]

Then:

\[ A_{\alpha 0} \xrightarrow{\varphi_0} A_{\alpha 2} \xrightarrow{\varphi_1} A_{\alpha 4} \xrightarrow{\varphi_2} \ldots \]

converges to $C^*(\Gamma, \psi_\alpha)$, where $\Gamma = \mathbb{Q}_N^2$ and $\psi_\alpha$ is as defined previously.
**Twisted $\Gamma$-algebras as noncommutative solenoids**

**Notation:** Let $\Gamma = \mathbb{Q}_N^2$ and for fixed $\alpha \in \Xi_N$, let $\psi_\alpha$ be the multiplier on $\Gamma$ defined earlier. Henceforth we denote the twisted group $C^*$-algebra $C^*(\Gamma, \psi_\alpha)$ by

$$A^S_\alpha$$

and call the $C^*$-algebra $A^S_\alpha$ a **noncommutative solenoid**.
Noncommutative solenoids have been written as direct limits of noncommutative tori. Since the $K$-theory of noncommutative tori is well-known, and since $K$-theory for $C^*$-algebras behaves well under direct limits, we compute the $K$-theory of our $C^*$-algebras $A^S_\alpha$ in a straightforward fashion.

Recall that for $\theta \in (0, 1)$, $K_0(A_\theta) \cong \mathbb{Z}^2$ is generated by the classes of the identity and a Rieffel projection $P$ of trace $\theta$, which we denote by $(1, 0)$ and $(0, 1)$ respectively. We also know that $K_1(A_\theta) \cong \mathbb{Z}^2$ is generated by the classes of $U$ and $V$, denoted respectively by $(1, 0)$ and $(0, 1)$. 
K-groups via direct limits

It follows from the continuity of $K$-groups that we can calculate $K_0$ and $K_1$ for by taking the direct limits of the following sequences of discrete abelian groups, respectively:

$$
K_0(A_{\alpha_0}) \xrightarrow{(\varphi_0)_*} K_0(A_{\alpha_2}) \xrightarrow{(\varphi_1)_*} K_0(A_{\alpha_4}) \xrightarrow{(\varphi_2)_*} \cdots
$$

and

$$
K_1(A_{\alpha_0}) \xrightarrow{(\varphi_0)_*} K_1(A_{\alpha_2}) \xrightarrow{(\varphi_1)_*} K_1(A_{\alpha_4}) \xrightarrow{(\varphi_2)_*} \cdots
$$

Thus all comes down to computing the structure of $(\varphi_k)_*$ as a homomorphism from $K_0(A_{\alpha_{2k}})$ to $K_0(A_{\alpha_{2(k+1)}})$, and as a homomorphism from $K_1(A_{\alpha_{2k}})$ to $K_1(A_{\alpha_{2(k+1)}})$. 
Lemma

(L.-P.) Fix a positive integer $N > 1$, and let $\alpha \in \Xi_N$. For each $k \in \mathbb{N}$, let $\varphi_k : A_{\alpha 2^k} \to A_{\alpha 2^{(k+1)}}$ be the embeddings described earlier, and let

$K_0(\varphi_k) : K_0(A_{\alpha 2^k}) \to K_0(A_{\alpha 2^{(k+1)}})$ and
$K_1(\varphi_k) : K_1(A_{\alpha 2^k}) \to K_1(A_{\alpha 2^{(k+1)}})$

be the induced morphisms on $K$-theory. Then

$K_0(\varphi_k)((1,0)) = (1,0), \ K_0(\varphi_k)((0,1)) = (r_\alpha^k, N),$
$K_1(\varphi_k)((1,0)) = (N,0), \ and \ K_1(\varphi_k)((0,1)) = (0, N),$

where $r_\alpha^k$ is the unique integer such that $N^2 \alpha 2^{k+2} = \alpha 2^k + r_\alpha^k$. 
Recall that for an abelian group $D$, the group $\text{Ext}(D, \mathbb{Z})$ is defined as a set to be the set of equivalence classes of abelian extensions of $D$ by $\mathbb{Z}$. There is a canonical way to give $\text{Ext}(D, \mathbb{Z})$ a group structure. We need knowledge of $\text{Ext}(\mathbb{Q}_N, \mathbb{Z})$ to calculate the $K_0$-group of $\mathcal{A}_\alpha$.

**Definition**

Let $N \in \mathbb{N}$, $N > 1$. Set:

$$\mathbb{Z}_N = \{(J_k)_{k \in \mathbb{N}} : J_0 = 0, \forall k \in \mathbb{N}, \exists j \in \{0, \cdots, N-1\} J_{k+1} = J_k + N^k j\}.$$ 

This set is made into a group with the following operation. If $J, K \in \mathbb{Z}_N$ then $J + K$ is the sequence $(L_k)_{k \in \mathbb{N}}$ where $L_k$ is the remainder of the Euclidean division of $J_k + K_k$ by $N^k$ for all $k \in \mathbb{N}$, $k > 0$. We call $\mathbb{Z}_N$ the group of $N$-adic integers.
A calculation involving the Cartan-Eilenberg long exact sequence in $\text{Ext}$ theory shows that there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \overset{i}{\longrightarrow} \mathbb{Z}_N \longrightarrow \text{Ext}(\mathbb{Q}_N, \mathbb{Z}) \longrightarrow 0.$$

where $\mathbb{Z}_N$ is as defined above:
We embed $\mathbb{Z}$ of integers into $\mathbb{Z}_N$ as follows: If $n$ is a positive integer, let $\iota(n)_k$ be the remainder for the division of $n$ by $N^k$, and $\iota(-n)$ is defined to be the additive inverse of $\iota(n)$.

We note that $\mathbb{Z}_N$ can also be identified with

$$\prod_{i=0}^{\infty}\{0, 1, \cdots, N-1\}_i = \{(j_i)_{i=0}^{\infty} : j_i \in \{0, 1, \cdots, N-1\}\}$$

via the correspondence

$$J_0 = 0, J_k = \sum_{i=0}^{k-1} j_i N^i, \ k \geq 1.$$ 

Thus $\mathbb{Z}_N$ has the structure of a compact abelian group.
The extension of $Q_N$ corresponding to $J \in \mathbb{Z}_N$:

A calculation: Let $J = (J_k) \in \mathbb{Z}_N$. We define

$$\xi_J : Q_N \times Q_N \to \mathbb{Z}$$

by

$$\xi_J \left( \frac{p_1}{N^{k_1}}, \frac{p_2}{N^{k_2}} \right) =
\begin{cases}
- \frac{p_1}{N^{k_1}} (J_{k_2} - J_{k_1}) & \text{if } k_2 > k_1 \\
- \frac{p_2}{N^{k_2}} (J_{k_1} - J_{k_2}) & \text{if } k_1 > k_2 \\
\frac{q}{N^r} (J_{k_1} - J_r) & \text{if } \land \left\{ k_1 = k_2, \frac{p_1}{N^{k_1}} + \frac{p_2}{N^{k_2}} = \frac{q}{N^r} \right\}
\end{cases}$$

Then $\xi_J$ is a $\mathbb{Z}$-valued symmetric 2-cocycle on $Q_N$, and $\xi_J_1$ is cohomologous to $\xi_J_2$ if and only if there exists $z \in \mathbb{Z}$ such that $J_1 = J_2 + \nu(z)$. 
The $K_0$ group of $A^S_{\alpha}$

**Theorem**

(L.-P.) Let $N \in \mathbb{N}$, $N > 1$, and fix $\alpha \in \Xi_N$. Define $J^\alpha \in \mathbb{Z}_N$ by

$$J^\alpha_k = N^k \alpha_k - \alpha_0, \ k \geq 1.$$ 

Define the central extension $Q_\alpha$ of $Q_N$ by $\mathbb{Z}$ as follows: as a set, $Q_\alpha = Q_N \times \mathbb{Z}$ and

$$(\frac{p_1}{N^{k_1}}, z_1) \cdot (\frac{p_2}{N^{k_2}}, z_2) = (\frac{p_1}{N^{k_1}} + \frac{p_2}{N^{k_2}}, z_1 + z_2 + \xi J^\alpha(\frac{p_1}{N^{k_1}}, \frac{p_2}{N^{k_2}})).$$

Then

$$K_0(A^S_{\alpha}) \cong Q_\alpha, \text{ and } K_1(A^S_{\alpha}) \cong Q_N^2.$$
**The range of the trace on $K_0$**

**Remark:** Recall that $\mathcal{A}_\alpha^S = C^*(\Gamma, \psi_\alpha)$, and that the dual group of $\Gamma$, $S_N \times S_N$, has a natural dual action on $C^*(\Gamma, \psi_\alpha)$. This gives a tracial state $\tau$ on $C^*(\Gamma, \psi_\alpha)$ that is invariant under the action of $S_N \times S_N$. If we consider $\mathcal{A}_\alpha^S$ as a direct limit of rotation algebras, the trace restricted to any subalgebra that is a rotation algebra is well-known. This allows us to compute the range of the trace on $K_0$:

**Corollary**

(L.-P.) Let $N \in \mathbb{N}$, $N > 1$, and fix $\alpha \in \Xi_N$. Let $\tau$ be any tracial state on $\mathcal{A}_\alpha^S$ (if $\mathcal{A}_\alpha^S$ is simple there is only one). Then the range of the trace $\tau$ on $K_0(\mathcal{A}_\alpha^S)$ is the subgroup of $\mathbb{R}$ generated by the integers $\mathbb{Z}$ and

$$\{\alpha_0, \alpha_1, \alpha_2, \cdots\}.$$
The range of the trace on $K_0$

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$$\{\alpha_0, \alpha_1, \alpha_2, \cdots, \}.$$
Proof: This follows from our description of $K_0(A^S_{\alpha})$ as a direct limit of $K_0$ groups of rotation algebras, and our knowledge of the values the trace takes at each stage on Rieffel projections.
We have a general theorem regarding classification, but it is easiest to state for prime values of $N$:

**Theorem**

(L.-P.) Let $N, M \in \mathbb{N}$ be prime numbers. Let $\alpha \in \Xi_N$ and $\beta \in \Xi_M$. The following assertions are equivalent:

1. The C*-algebras $A^S_\alpha$ and $A^S_\beta$ are *-isomorphic
2. We have $N = M$ and one of the sequences $\alpha$ and either $\beta$ or $-\beta = (1 - \beta_n)_{n \in \mathbb{N}}$ is a truncated subsequence of the other.

**Remark:** By saying that $\alpha$ is a truncated subsequence of $\beta$, we mean that there exists $k \in \mathbb{N} \cup \{0\}$ such that $\alpha_n = \beta_{n+k}, \forall n \in \mathbb{N}$.
Another corollary regarding \( AT \) algebras

**Another observation:**

**Corollary**

Let \( N \in \mathbb{N}, \ N > 1, \) and fix \( \alpha \in \Xi_N. \) If \( \alpha_0 \notin \mathbb{Q}, \) (or equivalently, if there exists \( n \in \mathbb{N} \) such that \( \alpha_n \notin \mathbb{Q} \)), then \( \mathcal{A}_\alpha^S \) is a simple \( AT \)-algebra of real rank 0.

**Remark:** We already know about the simplicity, and the \( AT \) and real rank 0 result follows from \( \mathcal{A}_\alpha^S \) being a direct limit of such algebras.
Projective modules for $\mathcal{A}_\alpha$ and higher dimensional non-commutative tori are understood by work of M. Rieffel, and F. Luef has extended this work to build modules with a dense subspace of functions coming from modulation spaces (e.g., Feichtinger’s algebra) with nice properties. What should be the approach to build projective modules over noncommutative solenoids? On approach is to build the projective modules from the “inside out”.
However there are some straightforward observations that can be made.

**Corollary**  

*Let* \( N \in \mathbb{N} \), \( N > 1 \), *and fix* \( \alpha \in \Xi_N \), *with* \( \alpha_0 \notin \mathbb{Q} \). *If* \( \gamma \in tr(K_0(A^S_{\alpha})) \) *with* \( \gamma > 0 \), *then there is a projective module over* \( A^S_{\alpha} \) *with trace* \( \gamma \).

**Remark:** Suppose \( \gamma = m + c_0 \alpha_0 + c_1 \alpha_1 + \cdots + c_n \alpha_n > 0 \) where \( m, c_1, c_2, \ldots, c_n \in \mathbb{Z} \). *Then* \( \gamma \in tr(K_0(A_{\alpha_n})) \) *for* \( A_{\alpha_n} \) *a subalgebra of* \( A^S_{\alpha} \) *coming from the direct limit construction. *Work of Rieffel then shows that there is a projective module over* \( A_{\alpha_n} \) *of trace* \( \gamma \) *and this gives the desired projective module over* \( A^S_{\alpha} \).
Morita equivalences between noncommutative solenoids?

Example

A simple scaling argument shows that unlike in the irrational rotation algebra case, if $\alpha, \beta \in \Xi_N$ are such that

$$\beta_0 = \frac{1}{\alpha_0},$$

it is not possible to have $A^S_\alpha$ strongly Morita equivalent to $A^S_\beta$ in general. This is because there does not exist $r \in \mathbb{R}$ with

$$r \cdot tr(K_0(A^S_\alpha)) = tr(K_0(A^S_\beta)).$$
Directed Systems of Equivalence Bimodules

Suppose we have two directed sequences of $C^*$-algebras

$$A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \ldots$$

and

$$B_0 \xrightarrow{\psi_0} B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} \ldots$$

Suppose further that for each $n \in \mathbb{N}$ there is an equivalence bimodule $X_n$ between $A_n$ and $B_n$

$$A_n - X_n - B_n,$$

and that the $\{X_n\}$ form a directed system.
That is, suppose there exists a direct system of module monomorphisms

\[ X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \ldots \]

satisfying

\[ \langle i_n(f), i_n(g) \rangle_{B_{n+1}} = \psi_n(\langle f, g \rangle_{B_n}), \quad f, g \in X_n, \text{ and} \]

\[ i_n(f \cdot b) = i_n(f) \cdot \psi_n(b), \quad f \in X_n, \ b \in B_n, \]

with analogous but symmetric equalities holding for the \( X_n \) viewed as left-\( A_n \) modules. Then letting \( \mathcal{A} \) be the direct limit of the \( A_n \), \( \mathcal{B} \) the direct limit of the \( B_n \), and \( \mathcal{X} \) the direct limit of the \( X_n \), (completed in the \( C^* \)-module norm), \( \mathcal{X} \) is an \( A - B \) equivalence bimodule.
Directed Systems of Equivalence Bimodules, continued

That is, suppose there exists a direct system of module monomorphisms

\[ X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \ldots \]

satisfying

\[ \langle i_n(f), i_n(g) \rangle_{B_{n+1}} = \psi_n(\langle f, g \rangle_{B_n}), \quad f, g \in X_n, \text{ and} \]
\[ i_n(f \cdot b) = i_n(f) \cdot \psi_n(b), \quad f \in X_n, \quad b \in B_n, \]

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Morita equivalence example:

I discuss a very simple example, to discuss how the directed system of bimodules is constructed.

**Example**

Fix irrational \( \alpha_0 \in [0, 1) \), let \( N = 2 \), and consider \( \alpha \in \Xi \) given by

\[
\alpha = (\alpha_0, \frac{\alpha_0}{2}, \frac{\alpha_0}{4}, \ldots, \frac{\alpha_n}{2^n}, \ldots,)
\]

Consider \( p_{\alpha_0} \in A_{\alpha_0} \subset A_{\alpha_1} \) a projection of trace \( \alpha_0 = 2\alpha_1 \). The bimodule

\[
A_{\alpha_0} - A_{\alpha_0} \cdot p_{\alpha_0} - p_{\alpha_0} A_{\alpha_0} p_{\alpha_0}
\]

is equivalent to Rieffel’s bimodule

\[
A_{\alpha_0} - \mathcal{C}_c(\mathbb{R}) - A_{\frac{1}{\alpha_0}} = B_0.
\]
Morita equivalence example, continued:

**Example**

**Continuation of example: part 2**

Let $\beta_0 = \frac{1}{\alpha_0}$. Rieffel’s theory again shows

$A_{\alpha_1} - A_{\alpha_1} \cdot p_{\alpha_0} - p_{\alpha_0}A_{\alpha_1}p_{\alpha_0}$

is the same as

$A_{\alpha_1} - A_{\alpha_1} \cdot p_{2\alpha_1} - p_{2\alpha_1}A_{\alpha_1}p_{2\alpha_1}$

which is equivalent to Rieffel’s bimodule

$A_{\alpha_1} - C_c(\mathbb{R} \times F_2) - C(\mathbb{T} \times F_2) \rtimes_{\tau_1} \mathbb{Z} = B_1$,

where $F_2 = \mathbb{Z}/2\mathbb{Z}$, and the action of $\mathbb{Z}$ on $\mathbb{T} \times F_2$ is given by multiples of $(\frac{\beta_1}{2}, [1]_{F_2})$, for $\beta_1 = \frac{1}{\alpha_1}$, i.e. multiples of $(\frac{1}{\alpha_0}, [1]_{F_2})$, i.e. multiples of $(\beta_0, [1]_{F_2})$. 
Example

Continuation of example: part 3

At the \( n^{th} \) stage we have
\[
A_{\alpha_n} - A_{\alpha_n} \cdot p_{\alpha_0} - p_{\alpha_0} A_{\alpha_n} p_{\alpha_0}
\]
is the same as
\[
A_{\alpha_n} - A_{\alpha_n} \cdot p_{2^n \alpha_n} - p_{2^n \alpha_n} A_{\alpha_n} p_{2^n \alpha_n}
\]
which is equivalent to
\[
A_{\alpha_n} - C_c(\mathbb{R} \times F_{2^n}) - C(\mathbb{T} \times F_{2^n}) \rtimes \tau_n \mathbb{Z} = B_n,
\]
where the action of \( \mathbb{Z} \) on \( \mathbb{T} \times F_{2^n} \) is given by multiples of \( (\frac{\beta_n}{2^n}, [1]_{F_{2^n}}) \), for \( \beta_n = \frac{1}{\alpha_n} = \frac{2^n}{\alpha_0} \), i.e. multiples of \( (\frac{1}{\alpha_0}, [1]_{F_{2^n}}) \), i.e. multiples of \( (\beta_0, [1]_{F_{2^n}}) \), for \( F_{2^n} = \mathbb{Z}/2^n\mathbb{Z} \).
Morita equivalence example, continued:

Example

Completion of example: part 4
From calculating the embeddings, we see that for
\( \alpha = (\alpha_0, \frac{\alpha_0}{2}, \cdots, \frac{\alpha}{2^n}, \cdots) \in \Xi_2, \) we have

\( \mathcal{A}_\alpha^S \) is strongly Morita equivalent to a direct limit \( \mathcal{B} \) of the \( B_n \)
(whose structure is not so easy to describe).
As expected, one calculates

\[
\text{tr}(K_0(\mathcal{A}_\alpha^S)) = \alpha_0 \cdot \text{tr}(K_0(\mathcal{B})).
\]
Summary

Main Results:
- Cohomology classes of multipliers on $\mathbb{Q}^2_N$ are in 1–1 correspondence with elements of $S_N$, and give twisted group $C^*$-algebras $C^*(\mathbb{Q}^2_N, \psi_\alpha) = A^S_\alpha$ that we call noncommutative solenoids.
- Necessary and sufficient conditions on $\psi_\alpha$ for these $C^*$-algebras to be simple are given.
- We describe $A^S_\alpha$ as direct limits of rotation algebras, and calculate their $K$-groups, which involves $\text{Ext}(\mathbb{Q}^2_N, \mathbb{Z})$.
- We compute the range of a tracial state on $K_0$, obtain a classification result, and discuss an example involving Morita equivalence.
Main Results:

- Cohomology classes of multipliers on $\mathbb{Q}_N^2$ are in 1–1 correspondence with elements of $S_N$, and give twisted group $C^*$-algebras $C^*(\mathbb{Q}_N^2, \psi_\alpha) = A_\alpha^S$ that we call noncommutative solenoids.

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References: