On approximation properties of Kantorovich-type sampling operators

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The aim of this talk is to study approximation properties of a generalized Shannon sampling operators

\[(S_w f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right)s(wt - k)\]

with a bandlimited kernel \(s\), which is a Fourier transform of a certain even window function \(\lambda \in C_{[-1,1]}\), \(\lambda(0) = 1\), \(\lambda(u) = 0\) (\(|u| \geq 1\)), i.e.

\[s(t) := s(\lambda; t) := \int_{0}^{1} \lambda(u) \cos(\pi tu) \, du.\]

We give the conditions for the kernel \(s\), which allow us to have the estimate of order of approximation via modulus of smoothness in form:

\[\|f - S_w^r f\| \leq M_r \omega_{2r}(f; \frac{1}{w}).\]
Def

The Bernstein class $B^p_\sigma$ for $\sigma \geq 0$ and $1 \leq p \leq \infty$ consists of those bounded functions $f \in L^p(\mathbb{R})$ which can be extended to an entire function $f(z)$ ($z \in \mathbb{C}$) of exponential type $\sigma$, i.e.,

$$|f(z)| \leq e^{\sigma|y|} \|f\|_C \quad (z = x + iy \in \mathbb{C}).$$

The class $B^p_\sigma$ is a Banach space if one takes the norm of $L^p(\mathbb{R})$.

Lemma

We have

$$B^1_\sigma \subset B^p_\sigma \subset B^r_\sigma \subset B^\infty_\sigma, \quad 1 \leq p \leq r \leq \infty,$$

$$B^p_\alpha \subset B^p_\beta, \quad 0 \leq \alpha \leq \beta < \infty.$$
Theorem (Whittaker-Kotelnikov-Shannon)

If \( g \in B^p_{\pi W}, 1 \leq p < \infty, \) or \( g \in B^\infty_\sigma \) for some \( 0 \leq \sigma < \pi W, \) then

\[
g(t) = \sum_{k=-\infty}^{\infty} g\left(\frac{k}{W}\right) \text{sinc}(Wt - k) =: (S^{\text{sinc}}_W g)(t),
\]

(1)

the series being uniformly convergent on each compact subset of \( \mathbb{R}. \)

The Bernstein class \( B^p_{\pi W} \) forms the set of fixed points of the sampling operators (1). Now it is natural to ask what happens in WKS Theorem if instead of \( g \in B^p_{\pi W} \) we consider uniformly continuous and bounded functions \( f \in C(\mathbb{R})? \) M. Theis [Theis’19] showed that the equality in (1) does not hold for any \( g \in C(\mathbb{R}). \)
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Generalized Shannon sampling series

We get uniform convergence

$$\| f - S_w f \|_C \to 0 \quad (w \to \infty)$$

for any $f \in C(\mathbb{R})$ if, instead of the cardinal sine, we use different kernel functions $s \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ (sinc $\not\in L^1(\mathbb{R})$), $\sum_{k \in \mathbb{Z}} s(u - k) = 1$, $u \in \mathbb{R}$. This approach gives us the generalized sampling series for $t \in \mathbb{R}$, $w > 0$

$$(S_w f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right)s(\omega t - k). \quad (2)$$

Whereas a generalized sampling series with a particular kernel were already considered by M. Theis in 1919, a study of those series for arbitrary kernel functions was initiated at RWTH Aachen University (P. L. Butzer et al) and widely studied there since 1977 and in Perugia since 2005.
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Take a function $f$ and compute the corresponding sampling series

$$(S_2 f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2}\right) s(2t - k), \quad (W = 2)$$

and the approximation error $S_2 f - f$. 
Now we compute the sampling series, taking \( W = 6 \),

\[
(S_6 f)(t) := \sum_{k=-\infty}^{\infty} f \left( \frac{k}{6} \right) s(6t - k)
\]

and the approximation error \( S_6 f - f \).
At last we take $W = 16$, getting

$$(S_{16}f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{16}\right) s(16t - k)$$

and the approximation error $S_{16}f - f$. 
Since in practice signals are however often discontinuous, this talk is also concerned with the convergence of $S_wf$ to $f$ in the $L^p(\mathbb{R})$-norm for $1 \leq p < \infty$.

The sampling series $S_wf$ of an arbitrary $L^p$-function $f$ may be divergent. Take $f \in L^p(\mathbb{R})$, $f(k/w) = 1$ for a fixed $w > 0$, then $(S_wf)(t) \equiv 1 \not\in L^p(\mathbb{R})$.

Also operators $S_wf$ do not always provide good results in applications, e.g. in Signal Processing. Generalized sampling operators depend on exact values $f(k/w)$, whilst in practice we often have to deal with the so called "jitter errors" i.e. the impossibility to determine the exact values at the nodes. Replacing the exact value $f(k/w)$ with an average of $f$ around $k/w$ might lead to smaller errors and therefore be useful in applications.
Z. Burinska, K. Runovski and H. J. Schmeisser introduced in [BRS’06] for $0 < p \leq \infty$ additional shifts $\lambda$:

\[
(S_{w;\lambda} f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w} + \lambda\right)s\left(w(t - \lambda) - k\right)
\]

and estimated the order of approximation via modulus of smoothness.

C. Bardaro, P. L. Butzer, R. Stens and G. Vinti in [BBSV’06] introduced a suitable subspace of $L^p(\mathbb{R})$. They estimated the order of approximation for the classical (Whittaker-Kotel’nikov-Shannon) operator for $1 < p < \infty$ and for sampling operators with time-limited kernels for $1 \leq p < \infty$ in [Butzer, Stens’08] and [BBSV’10] via averaged modulus of smoothness.

We used the same approach and estimated the order of approximation for sampling operators with band-limited kernels for $1 < p < \infty$ in [T’13] via the averaged modulus of smoothness and for $1 \leq p < \infty$ in [Kivinukk,T’14] and [T’14] via the classical modulus of smoothness.
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C. Bardaro, P. L. Butzer, R. Stens and G. Vinti in [BBSV’07] introduced Kantorovich-type sampling operators

\[(S^K_w f)(t) = \sum_{k=-\infty}^{\infty} \left( \frac{(k+1)/w}{w} \int_{k/w}^{(k+1)/w} f(u) \, du \right) s(wt - k)\]

and proved the convergence for $1 \leq p < \infty$ case.

We estimated the order of approximation for Kantorovich-type sampling operators

\[(S^l_{w,n} f)(t) = \sum_{k=-\infty}^{\infty} \left( \frac{(2nk+1)/2nw}{nw} \int_{(2nk-1)/2nw}^{(2nk+1)/2nw} f(u) \, du \right) s(wt - k)\]

with band-limited kernels for $1 \leq p \leq \infty$ in [Orlova,T’14] via the classical modulus of smoothness.
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\[(S^K_w f)(t) = \sum_{k=-\infty}^{\infty} \left( w \int_{k/w}^{(k+1)/w} f(u) \, du \right) s(wt - k) \]

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We estimated the order of approximation for Kantorovich-type sampling operators

\[(S^l_{w,n} f)(t) = \sum_{k=-\infty}^{\infty} \left( nw \int_{(2nk-1)/2nw}^{(2nk+1)/2nw} f(u) \, du \right) s(wt - k) \]

with band-limited kernels for \(1 \leq p \leq \infty\) in [Orlova,T’14] via the classical modulus of smoothness.
Singular integrals

In order to proceed with Kantorovich-type sampling operators we need to define the singular integral. If $\chi \in L^1(\mathbb{R})$ is such that

$$\int_{-\infty}^{\infty} \chi(u) du = 1,$$

then the convolution integral of $f$ with $\chi_\rho(u) := \rho \chi(\rho u)$, namely

$$ (l^\chi f)(x) := (f * \chi_\rho)(x) = \int_{-\infty}^{\infty} f(u) \rho \chi(\rho(x - u)) du \quad (x \in \mathbb{R})$$

is called the singular (convolution) integral with kernel $\chi$. 
As a final step we replace the exact value $f(k/w)$ in (2) with an average of $f$ around $k/w$, using the singular integral $I_{nw}^{\chi} f$ ($n \in \mathbb{N}$) with $k/w$ as an argument. Thus for $f \in L^p(\mathbb{R})$ ($1 \leq p \leq \infty$) the Kantorovich-type sampling operators are given by ($t \in \mathbb{R}; w > 0; n \in \mathbb{N}$)

\[
(S^K_{w,n} f)(t) := \sum_{k=-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(u) nw\chi(nw(\frac{k}{w} - u))du \right) s(wt - k) \quad (4)
\]

with kernels $s, \chi \in L^1(\mathbb{R}), \int_{-\infty}^{\infty} \chi(u)du = 1, \sum_{k \in \mathbb{Z}} s(u - k) = 1$ ($u \in \mathbb{R}$).

**Remark.** Bardaro, Butzer, Stens and Vinti considered the case $\chi = \chi[0,1], n = 1$ only.
Bandlimited kernels

In this talk we consider an even band-limited kernel $s$, i.e. $s \in B_{\pi}^1$, defined by an even window function $\lambda \in C_{[-1,1]}$, $\lambda(0) = 1$, $\lambda(u) = 0$ ($|u| \geq 1$) by the equality

$$s(t) := s(\lambda; t) := \int_{0}^{1} \lambda(u) \cos(\pi tu) \, du. \quad (5)$$

In fact, this kernel is the Fourier transform of $\lambda \in L^1(\mathbb{R})$,

$$s(t) = \sqrt{\frac{\pi}{2}} \lambda^{\wedge}(\pi t). \quad (6)$$

Lemma (Butzer, Splettstößer, Stens’88)

If $s \in B_{\pi}^1$, then for $f \in C(\mathbb{R})$

$$\| S_w^s f - f \|_C \asymp \| l_w^s f - f \|_C.$$
Many kernels can be defined by

\[ s(t) := \int_{0}^{1} \lambda(u) \cos(\pi tu) \, du, \]

e.g.
1) \( \lambda(u) = 1 \) defines the sinc function;
2) \( \lambda(u) = 1 - u \) defines the Fejér kernel \( s_F(t) = \frac{1}{2} \text{sinc} \frac{2t}{2} \) (cf. [13]);
3) \( \lambda_j(u) := \cos \pi(j + 1/2)u, j = 0, 1, 2, \ldots \) defines the Rogosinski-type kernel (see [7]) in the form

\[ r_j(t) := \frac{1}{2} \left( \text{sinc}(t + j + \frac{1}{2}) + \text{sinc}(t - j - \frac{1}{2}) \right) \]  

(7)
4) \( \lambda_H(u) := \cos^2 \frac{\pi u}{2} = \frac{1}{2}(1 + \cos \pi u) \) defines the Hann kernel (see [8])

\[ s_H(t) := \frac{1}{2} \frac{\text{sinc} t}{1 - t^2}; \]  

(8)
5) the general cosine window

\[ \lambda_{C,a}(u) := \sum_{k=0}^{m} a_k \cos k\pi u \]  \hspace{1cm} (9)

defines the Blackman-Harris kernel (see [9])

\[ s_{C,a}(t) := \frac{1}{2} \sum_{k=0}^{m} a_k \left( \text{sinc}(t - k) + \text{sinc}(t + k) \right) \]  \hspace{1cm} (10)

provided (here and following \( \lfloor x \rfloor \) is the largest integer less than or equal to \( x \in \mathbb{R} \))

\[ \sum_{k=0}^{\lfloor m/2 \rfloor} a_{2k} = \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} a_{2k-1} = \frac{1}{2}. \]  \hspace{1cm} (11)

We get Hann kernel (8) if we take \( m = 1 \) in (10).
6) powers of the Hann window (see [6], formula(25a))

\[
\lambda_{H,m}(u) := \cos^m \left( \frac{\pi u}{2} \right) \quad (12)
\]

\[
= \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} \cos \left( (k - \frac{m}{2}) \pi u \right), \quad (13)
\]

give a general Hann kernel in the form

\[
s_{H,m}(t) = 2^{-m} \frac{\Gamma(1 + m)}{\Gamma(1 + \frac{m}{2} - t)\Gamma(1 + \frac{m}{2} + t)}. \quad (14)
\]

Comparing the window function \( \lambda_{H,m} \) in (13) and the general cosine window \( \lambda_{C,a} \) in (9) we see that the general Hann kernel in case of \( m = 2n \) \((n \in \mathbb{N})\) is a special case of the Blackman-Harris kernel. Indeed, \( s_{H,2n} = s_{C,a^*} \), where the parameter vector \( a^* \in \mathbb{R}^{n+1} \) has components \( a_0^* = \frac{1}{2^{2n}} \binom{2n}{n} \) and \( a_k^* = \frac{1}{2^{2n-1}} \binom{2n}{n-k} \) for \( k = 1, 2, \ldots, n \).
7) the general Rogosinski-type window

\[
\lambda_{R,a}(u) := \sum_{j=0}^{m} a_j \lambda_j(u) = \sum_{j=0}^{m} a_j \cos \pi(j + 1/2)u
\] (15)

defines the general Rogosinski-type kernel

\[
r_a(t) := \frac{1}{2} \sum_{j=0}^{m} a_j \left( \text{sinc}(t - \frac{2j + 1}{2}) + \text{sinc}(t + \frac{2j + 1}{2}) \right)
\] (16)

provided

\[
\sum_{j=0}^{m} a_j = 1.
\] (17)

Comparing the window function \( \lambda_{H,m} \) in (13) and the general Rogosinski-type window \( \lambda_{R,a} \) in (15) we see that the general Hann kernel in case of \( m = 2n + 1 \) \((n \in \mathbb{N}_0)\) is a special case of the general Rogosinski-type kernel. Indeed, \( s_{H,2n+1} = s_{R,a^*} \), where the parameter vector \( a^* \in \mathbb{R}^{n+1} \) has components \( a^*_j = \frac{1}{2^{2n}} \binom{2n+1}{n-j} \) for \( j = 0, 1, \ldots, n \).
Kernels for singular integrals

In this talk we consider for the singular integrals an even kernel \( \chi \in L^1(\mathbb{R}) \) with absolute moment

\[
m_0(\chi) := \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\chi(u - k)| < \infty,
\]

defined by an even window function \( \mu \), \( \mu(0) = 1 \) by the equality

\[
\chi(t) := \chi(\mu; t) := \int_0^\infty \mu(u) \cos(\pi tu) \, du.
\]  

(18)

In fact, this kernel is also a kernel for generalized sampling operators, when we additionally require \( \mu(2k) = 0, \ k \in \mathbb{N} \).
Many kernels can be defined by

$$\chi(t) := \chi(\mu; t) := \int_{0}^{\infty} \mu(u) \cos(\pi tu) \, du$$

e.g.

1) $$\mu(u) = \text{sinc}^k \left( \frac{u}{2} \right) \quad (k \in \mathbb{N})$$ defines the B-spline kernel $$B_{k-1}$$, in particular case $$k = 1$$ we get the indicator function $$\chi[-1/2,1/2]$$;

2) $$\mu_a(u) = \sum_{k=0}^{m-1} a_k \text{sinc}^{k+1} \left( \frac{u}{2} \right)$$ defines a spline kernel of degree $$m$$ if $$\sum a_k = 1$$;

3) $$\mu(u) = e^{-\pi \left( \frac{u}{2} \right)^2}$$ defines a Gauss-Weierstraß kernel

$$\chi_{GW}(t) = e^{-\pi t^2}.$$
Theorem

For Kantorovich-type sampling operator $S^K_{w,n} = S_w^s(I_{nw}^\chi)$ we have for $1 \leq p \leq \infty$ an estimate of the operator norm $(1/p + 1/q = 1)$

$$\|S^K_{w,n}\|_{p \to p} \leq \left( n\|s\|_1 m_0(\chi) \right)^{1/p} \left( \|\chi\|_1 m_0(s) \right)^{1/q}$$

Remark. We see that in the case of generalized sampling operators $S_w$ (i.e. $n \to \infty$) this estimate is not bounded for $1 \leq p < \infty$.

Remark. Take $\chi = \chi[-1/2,1/2]$. We proved that in this case for $L^1(\mathbb{R})$ we have $\|S^K_{w,n}\|_{1 \to 1} = n\|s\|_1$ and for $C(\mathbb{R})$ we have $\|S^K_{w,n}\|_{C \to C} = m_0(s)$. 
**Nikolskii’s inequality**

**Theorem (Nikolskii’s inequality)**

Let $1 \leq p \leq \infty$. Then, for every $s \in B^p_\sigma$,

$$\|s\|_p \leq \sup_{u \in \mathbb{R}} \left\{ \sum_{k=-\infty}^{\infty} |s(u - k)|^p \right\}^{1/p} \leq (1 + \sigma)\|s\|_p.$$ 

Our kernels $s \in B^1_\pi$. By Nikolskii’s inequality we have

$$\|s\|_1 \leq m_0(s) = \|S_w\|,$$

which gives the estimate

$$\|S_{w,n}^K\|_{p \to p} \leq \|S_w\| \left( n m_0(\chi) \right)^{\frac{1}{p}} \|\chi\|_{\frac{1}{q}}$$
Theorem

If we have for sampling operator $S^s_w$

$$\| S^s_w f - f \|_p \leq M_1 \omega_k(f; \frac{1}{w})_p$$

and for the singular integral

$$\| I^x_{nw} f - f \|_p \leq M_2 \omega_\ell(f; \frac{1}{w})_p,$$

then for Kantorovich-type sampling operator $S^K_{w,n} = S^s_w(I^x_{nw})$

$$\| S^K_{w,n} f - f \|_p \leq M_3 \omega_r(f; \frac{1}{w})_p,$$

where $r := \min\{k, \ell\}$. 
Corollary

For Kantorovich-type sampling operator $S^K_{w,n} = S^s_w(I^\chi_{nw})$, where $s$ is Rogosinski-type, Hann, Blackman-Harris or B-spline kernel and $\chi$ is a indicator function or Gauss-Weierstraß kernel we have

$$\|S^K_{w,n}f - f\|_p \leq M\omega_2(f; \frac{1}{w})_p.$$ 

Indeed, for Rogosinski-type, Hann, Blackman-Harris and B-spline kernels we have the estimates of order of approximation for corresponding generalized sampling operators via modulus of smoothness order 2. For singular integrals with indicator function and Gauss-Weierstraß kernels we have the estimates of the order of approximation via modulus of smoothness order 2.
The main theorem

Let Kantorovich-type sampling operator $S^K_{w,n} = S^s_w(l^\infty_{nw})$ ($w > 0$, $n \in \mathbb{N}$) be defined by the kernel $s \in B^1_\pi$ with window $\lambda$ and by the kernel $\chi \in L^1$ with window $\mu$. If for some $r \in \mathbb{N}$

$$
\nu_n(u) := \lambda(u)\mu\left(\frac{u}{n}\right) = 1 - \sum_{j=r}^{\infty} c_j u^{2j}, \quad \sum_{j=r}^{\infty} |c_j| \leq \infty. \quad (19)
$$

Then for $f \in L^p(\mathbb{R})$ ($1 \leq p \leq \infty$)

$$
\|S^K_{w,n}f - f\|_p \leq M_r \omega_{2r}(f; \frac{1}{w})_p. \quad (20)
$$

The constants $M_r$ are independent of $f$ and $w$. 
Idea of the proof

Lemma (cf. Butzer, Stens’85)

If $g \in B_{\alpha \pi w}^p$ and $s \in B_{\beta \pi}^1$ ($\alpha + \beta = 2$), then

$$S_s^w g = I_s^w g.$$ 

Take $g \in B_{\pi w/n}^p$, then $I_{nw}^\chi g \in B_{\pi w}^p$ and for $S_{w,n}^K = S_w^s(I_{nw}^\chi)$

$$\| S_{w,n}^K f - f \|_p \leq \| S_{w,n}^K \|_{p \rightarrow p} \| f - g \|_p + \| I_w^s(I_{nw}^\chi g) - g \|_p + \| g - f \|_p.$$ 

$I_w^s(I_{nw}^\chi g) = s_w * \chi_{nw} * g = (s * \chi_n)_w * g = \varphi_w * g = I_w^\varphi g = S_w^\varphi g$

where $\varphi \in B_{\pi}^1$ is defined

$$\varphi(t) := \int_0^1 \nu_n(u) \cos \pi ut \, du = \int_0^1 \lambda(u) \mu \left( \frac{u}{n} \right) \cos \pi ut \, du.$$
The space $\Lambda^p$

Definition (Bardaro, Butzer, Stens, Vinti’ 2006)

Let $\Sigma := (x_j)_{j \in \mathbb{Z}} \subset \mathbb{R}$ be an admissible partition of $\mathbb{R}$, i.e.

$0 < \inf_{j \in \mathbb{Z}} \Delta_j \leq \sup_{j \in \mathbb{Z}} \Delta_j < \infty$, $\Delta_j := x_j - x_{j-1}$ and let the discrete $\ell^p(\Sigma)$-seminorm of a sequence of function values $f_\Sigma$ on $\Sigma$ of a function $f : \mathbb{R} \to \mathbb{C}$ be defined for $1 \leq p < \infty$ by

$$\|f\|_{\ell^p(\Sigma)} := \left\{ \sum_{j \in \mathbb{Z}} |f(x_j)|^p \Delta_j \right\}^{1/p}.$$ 

The space $\Lambda^p$ for $1 \leq p < \infty$ is defined by

$$\Lambda^p := \{ f \in \mathcal{M}(\mathbb{R}); \|f\|_{\ell^p(\Sigma)} < \infty \text{ for each admissible sequence } \Sigma \}.$$
Denote for $1 \leq p < \infty$ $X^p(\mathbb{R}) := \Lambda^p$ and $X^\infty(\mathbb{R}) := C(\mathbb{R})$. $B^p_\sigma \subset X^p(\mathbb{R})$.

**Theorem (Kivinukk, T’14)**

Let sampling operator $S^r_w$ ($w > 0$) be defined by the kernel $s_r \in B^1_\pi$ with $\lambda = \lambda_r$ and for some $r \in \mathbb{N}$ let

$$\lambda_r(u) := 1 - \sum_{j=r}^{\infty} c_j u^{2j}, \quad \sum_{j=r}^{\infty} |c_j| \leq \infty. \quad (21)$$

Then for $f \in X^p(\mathbb{R})$ ($1 \leq p \leq \infty$)

$$\|S^r_w f - f\|_p \leq M_r \omega_{2r}(f; \frac{1}{w})_p. \quad (22)$$

The constants $M_r$ are independent of $f$ and $w$. 
We have for $g \in B^p_{\frac{n}{w}}$

$$\| S^K_{w,n}g - g \|_p = \| S^\varphi_w g - g \|_p \leq M_r \omega_{2r}(g; \frac{1}{w})_p$$

$$\leq M_r \omega_{2r}(f; \frac{1}{w})_p + M_r \omega_{2r}(g - f; \frac{1}{w})_p \leq M_r \omega_{2r}(f; \frac{1}{w})_p + M_r 2^{2r} \| g - f \|_p.$$

**Theorem (Jackson-type theorem)**

Given $f \in C(\mathbb{R})$ or $f \in L^p(\mathbb{R})$ ($1 \leq p < \infty$). Then there exists a $g^*_\sigma \in B^p_\sigma$ ($1 \leq p \leq \infty$) and a constant $C_k > 0$ (depending only on $k \in \mathbb{N}$) such that

$$\| f - g^*_\sigma \|_p \leq C_k \omega_k(f; \frac{1}{\sigma})_p.$$
Examples

Theorem

Let \( C^{I}_{w,a} = S^s_w(I_{nw}^\chi) \) be a Kantorovich-type sampling operator whith \( s = s_{C,a} (a \in \mathbb{R}^{m+1}) \) and \( \chi = \chi[-1/2,1/2] \), then for \( f \in L^p(\mathbb{R}) \) (\( 1 \leq p \leq \infty \))

\[
\| C^{I}_{w,a,n}f - f \|_p \leq M_{a,2}\omega_2(f; \frac{1}{w})_p.
\]

Moreover, if there holds

\[
\sum_{k=1}^{m} a_k k^2 = -\frac{1}{(2^2 - 1)2^2 n^2},
\]

then

\[
\| C^{I}_{w,a,n}f - f \|_p \leq M_{a,4}\omega_4(f; \frac{1}{w})_p.
\]
The Bibliographic References I


Thank you!
[4], [13], [1], [2], [11], [5], [12], [10], [3]