\[ f(x) = \lim_{n \to \infty} f(x_n) \]

**Introduction**

Communicated by F. R. N. G. de Bruijn at the meeting of September 30, 1978

**MATHEMATICA**

By A. J. E. M. Jansen

**Convolutions Theory in a Space of Generalized Functions**

Communicated by F. R. N. G. de Bruijn at the meeting of September 30, 1978

**Deep of Mathematics, Foundation Turing of Technology**

By A. J. E. M. Jansen
The context within which the integration of $S$ is performed is crucial. The process of integrating $S$ and $\mathcal{A}$ gives rise to important conclusions. Specifically:

1. The integration of $S$ and $\mathcal{A}$ provides fundamental insights and conclusions.

2. The spaces $S$ and $\mathcal{A}$ are defined through specific operations.

3. The function $\phi$ plays a critical role in the overall integration process, affecting the outcome of the integration.

4. The operations $\mathcal{A}$ and $S$ are defined through specific expressions, which are integral to the overall process.

5. The integration of $S$ and $\mathcal{A}$ results in significant conclusions that are foundational to the field.

Overall, the integration of $S$ and $\mathcal{A}$ is a complex but crucial process that leads to profound conclusions.
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Now, let $\mathcal{D}(\mathbb{R}) := \{f \in \mathcal{S}(\mathbb{R}) | f(0) = 0 \}$, the space of compactly supported smooth functions.

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1.3. We proceed by defining convolution in $S^0$ and $\mathcal{S}(\mathbb{R})$. Let $\mathcal{S}(\mathbb{R})$ be a sequence in $S^0$, $S^0 \subset \mathcal{S}(\mathbb{R})$.
We introduce in this section convolution operators defined on $\mathbb{Z}$.

2.2. **PARAMETRIZATION**

$\mathbb{Z}^n = \mathbb{Z}$.

Note that $\mathbb{Z}$ is not a continuous linear space or that $\mathbb{Z}$ is not a continuous linear space.

$$\mathbf{(\mathbf{\xi} \in \mathbb{Z}^n)}$$

In this case, the convolution operator $\Delta^2$ is defined as $\Delta^2 \mathbf{\xi} = \mathbf{\xi} \otimes \mathbf{\xi}$.

Similarly, for $\mathbf{(\mathbf{\xi} \in \mathbb{Z}^n)}$, $\mathbf{\xi} \otimes \mathbf{\xi} = \mathbf{\xi} \otimes \mathbf{\xi}$.

The convolution operators are continuous on $\mathbb{Z}$.

There is a unique operator $\mathbb{Z}$, such that $\mathbf{\xi} \otimes \mathbf{\xi} = \mathbf{\xi} \otimes \mathbf{\xi}$.

For $\mathbf{(\mathbf{\xi} \in \mathbb{Z}^n)}$, $\mathbf{\xi} \otimes \mathbf{\xi} = \mathbf{\xi} \otimes \mathbf{\xi}$.

We denote the extended operator $\mathbb{Z}$ by $\mathbb{Z}$. For examples, see I.9.

**Proof.** This is [I.9], Theorem 2.2.

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3. CONVOLUTION OPERATIONS AND GENERALIZED FUNCTIONS

We now specialize our attention to operators of the form $T$ that map $x$ into $y$ and $y$ into $x$. With $\mu = \delta_0$ the convolution operator $T$ is defined by

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy$$

We shall denote the convolution of $f$ and $g$ by $f \ast g$

Definition: For $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, the convolution $f \ast g$ is defined by

$$(f \ast g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy$$

The convolution $f \ast g$ is a linear function of $f$ and $g$. The convolution $f \ast g$ is a convolution operator.

Theorem: If $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f \ast g \in L^1(\mathbb{R})$ and $f \ast g$ is continuous.

Proof: By the definition of convolution,

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is a linear function of $f$ and $g$. The convolution $f \ast g$ is a convolution operator.
\[f^\omega(\Delta L) \equiv^\omega f(\Delta L)\]

**Theorem:** Let \( F, G, H \) be \( \omega \)-sets, and \( \omega \) be an \( \omega \)-set. We have \( \Delta L - \Delta H \equiv^\omega f(\Delta L) \).

Another theorem of the above type is the following one:

The things are shown in general.

\[\Delta L - \Delta G = 0 \quad \text{if} \quad \Delta L - \Delta G \equiv^\omega 0\]

one by noting that \( \Delta L \) is shown in the above general case reduced to the above.

\[\Delta L - \Delta G = 0 \quad \text{if} \quad \Delta L - \Delta G \equiv^\omega 0\]

Proofs. First assume that \( \Delta L \equiv^\omega 0 \), and let \( \Delta G \equiv^\omega 0 \). We have

\[\Delta L - \Delta G = 0 \quad \text{if} \quad \Delta L - \Delta G \equiv^\omega 0\]

3. Theorem. Let \( F, G, H \) be \( \omega \)-sets. We have \( \Delta L - \Delta H \equiv^\omega f(\Delta L) \).

Proof. According to 1.4 we can extend \( \Delta L \) to \( \omega \). The extended operator \( \Delta L \) is shown in the above general case reduced to the above.

**Remark:** For the following relations hold:

\[\Delta L - \Delta G = 0 \quad \text{if} \quad \Delta L - \Delta G \equiv^\omega 0\]

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so that \( \Delta L - \Delta G \) are adjoint operators.

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Moreover, and the above hold by that holds by

\[\Delta L - \Delta G = 0 \quad \text{if} \quad \Delta L - \Delta G \equiv^\omega 0\]
Theorem 2. Let $\varphi$ be a continuous linear operator on the space $H$. Then, for every $\psi \in H$, we can define the convolution $\varphi * \psi$ of $\varphi$ and $\psi$.

Proof. We define $$(\varphi * \psi)(t) = \int_{-\infty}^{\infty} \varphi(t-s) \psi(s) \, ds$$
for every $t \in \mathbb{R}$. This integral exists because $\varphi$ is continuous and $\psi$ is bounded. It can be shown that $\varphi * \psi$ is also continuous and linear.

Remark 2. We mention the possibility of extending this definition to the space $L^p$ for $1 \leq p < \infty$.

Remark 3. We observe that the convolution operation is associative, that is, $$(\varphi * (\psi * \psi')) = ((\varphi * \psi) * \psi')$$
for every $\varphi, \psi, \psi' \in H$. It can be shown that this property holds for all $\varphi, \psi, \psi' \in L^p$.

Proof. We prove this property by showing that the convolution operation is associative. Let $\varphi, \psi, \psi' \in H$. Then, for every $t \in \mathbb{R}$,

$$(\varphi * (\psi * \psi'))(t) = \int_{-\infty}^{\infty} \varphi(t-s) (\psi * \psi')(s) \, ds$$

and

$$(\varphi * (\psi * \psi'))(t) = \int_{-\infty}^{\infty} \varphi(t-s) \psi(s) \psi'(s) \, ds$$

Since $\psi * \psi'$ is a continuous linear operator on $H$, it can be shown that $$(\varphi * (\psi * \psi'))(t) = \int_{-\infty}^{\infty} \varphi(t-s) \psi(s) \psi'(s) \, ds = ((\varphi * \psi) * \psi')(t)$$
for every $t \in \mathbb{R}$. This proves the associativity of the convolution operation.

Theorem 3. Let $\varphi$ be a continuous linear operator on the space $H$. Then, for every $\psi \in H$, we can define the convolution $\varphi * \psi$ of $\varphi$ and $\psi$.

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for every $t \in \mathbb{R}$. This integral exists because $\varphi$ is continuous and $\psi$ is bounded. It can be shown that $\varphi * \psi$ is also continuous and linear.

Remark 4. We mention the possibility of extending this definition to the space $L^p$ for $1 \leq p < \infty$.
Theorem 1. Let \( \mathcal{L} \) be a language. We have for every \( \mathcal{L} \) the following characterization of \( \mathcal{L} \) as a complete set of generalized functionals for \( \mathcal{L} \):

\[
\forall \mathcal{L} \text{ such that } \mathcal{L} \text{ is complete for } \mathcal{L}, \quad \mathcal{L} \text{ is a complete set of generalized functionals for } \mathcal{L}.
\]

The following proposition (once again, this is not unique, hence the mapping only) is well defined on \( \mathcal{L} \).

Proposition 1. Let \( \mathcal{L} \) be a language. We have for every \( \mathcal{L} \):

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\]

We also have the analogous proposition for \( \mathcal{L} \), which is again denoted by \( \mathcal{L} \).
Theorem 4.1 can be generalized as follows:

Let

\[ f(x) = \left\{ \begin{array}{ll}
1 & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{array} \right. \]

Then

\[ \int_A f(x) \, dx = \int f(x) \, dx = 1. \]

Moreover, this statement is true for any compact subset of \( \mathbb{R} \).
We first prove the formula with $\mathcal{A} \in \mathcal{B}$ and $\mathcal{F} \in \mathcal{G}$, which can be approached in two ways:

**Theorem.** If $\mathcal{A} \in \mathcal{B}$, then $\mathcal{A} \in \mathcal{B}_\mathcal{F} \iff \mathcal{A} \in \mathcal{B}_\mathcal{G}$.

*Proof.* We prove for every $\mathcal{A} \in \mathcal{B}_\mathcal{F}$, $\mathcal{A} \in \mathcal{B}_\mathcal{G}$.

Let $\mathcal{A} \in \mathcal{B}_\mathcal{F}$, then $\mathcal{F} \in \mathcal{B}_\mathcal{G}$, and $\mathcal{A} \in \mathcal{B}_\mathcal{G}$.

For $\mathcal{F} \in \mathcal{B}_\mathcal{G}$ and $\mathcal{A} \in \mathcal{B}_\mathcal{F}$, we have $\mathcal{A} \in \mathcal{B}_\mathcal{G}$.

The proof follows from the properties of $\mathcal{B}_\mathcal{G}$, and it follows from the properties of $\mathcal{B}_\mathcal{F}$.

**Theorem.** If $\mathcal{A} \in \mathcal{B}_\mathcal{G}$, then $\mathcal{A} \in \mathcal{B}_\mathcal{F}$.

*Proof.* Let $\mathcal{A} \in \mathcal{B}_\mathcal{G}$, then $\mathcal{A} \in \mathcal{B}_\mathcal{F}$.

This theorem is analogous to the previous one.

In the proofs of the above theorems, the convolution property is used in the particular case $\mathcal{B}_\mathcal{G} = \mathcal{B}_\mathcal{F}$, and it will turn out that $\mathcal{A} \in \mathcal{B}_\mathcal{G}$ implies $\mathcal{A} \in \mathcal{B}_\mathcal{F}$. This means that $\mathcal{A} \in \mathcal{B}_\mathcal{G}$.

In some particular remarks on convolution theory,

$$
\left(\mathcal{H}(0), \mathcal{U}, \mathcal{Y}, \mathcal{Z}\right) = \mathcal{H}(0), \mathcal{U}, \mathcal{Y}, \mathcal{Z}
$$

obtained the desired result. The proof is as follows:

Let $\mathcal{A} \in \mathcal{B}_\mathcal{G}$, then $\mathcal{A} \in \mathcal{B}_\mathcal{F}$.

The convolution property

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The convolution property

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$$

proves the theorem.
Let $\mathcal{L}$ be a continuous linear operator on $X$. Then, we have

$$\delta X \mathcal{L} = \delta^X \mathcal{L}.$$
Theorem 5.2. Let \( Y \in \mathbb{R}^m \) and \( Z \in \mathbb{R}^n \) be two-dimensional vectors. Then, for any \( \alpha \in \mathbb{R} \), the following holds:

\[
(\alpha Y + Z) = \alpha Y + Z
\]

Proof: By the two-dimensional version of the Cauchy-Schwarz inequality, we have

\[
\|\alpha Y + Z\| \leq \|\alpha Y\| + \|Z\|
\]

Equality holds if and only if \( Y \) and \( Z \) are linearly dependent.

Theorem 5.3. Let \( Y \in \mathbb{R}^m \) and \( Z \in \mathbb{R}^n \) be two-dimensional vectors. Then, for any \( \alpha \in \mathbb{R} \), the following holds:

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