The duality condition for Weyl-Heisenberg frames

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Abstract: We present formulations of the condition of duality for Weyl-Heisenberg systems in the time domain, the frequency domain, the Zak transform domain, and for rational time-frequency sampling factors. Many of these results are presented in a more general framework of shift-invariant time-continuous systems and characterize minimal dual systems.

1.1 Introduction

Let us start by explaining what we mean by a shift-invariant system, both for the time-continuous and the time-discrete case. Such a system consists of a collection of functions or sequences \( g_m \) of the form

\[
g_m = g_0(-n\Delta), \quad (n,m) \in \mathbb{Z} \times I,
\]

where \( g_m \in L^2(\mathbb{R}) \) or \( p^2(\mathbb{Z}) \), \( \Delta \in \mathbb{R}^+ \) or \( \Delta = 0 \) or \( \Delta = e_1 \in \mathbb{R}^d \) or \( \Delta = 0 \) or \( I = \mathbb{Z} \) or \( \Delta = e_1 \in \mathbb{R}^d \), \( I = \mathbb{Z} \) or \( I = \mathbb{Z} \times \mathbb{Z} \) is referred to as the time-continuous case while the case with \( g_m \in p^2(\mathbb{Z}) \), \( \Delta \in \mathbb{N}, \Delta = 0 \) or \( M = 1 \) is referred to as the time-discrete case. We do not consider systems that consist of functions or sequences \( g_m \) that are periodic.

We are interested in finding dual systems

\[
g_m = g_m(-n\Delta), \quad (n,m) \in \mathbb{Z} \times I.
\]
by which we mean that any \( f \in L^2(\mathbb{R}) \) or \( l^2(\mathbb{Z}) \) has an \( L^2(\mathbb{R}) \)-or \( l^2(\mathbb{Z}) \)-convergent representation
\[
f = \sum_{n,m} \langle f, \gamma_{nm} \rangle g_{nm} . \tag{1.1.3}
\]
For this to be meaningful we require the involved systems to have a finite frame upper bound: the system (1.1.1) has a finite frame upper bound when there is a \( B_g < \infty \) such that
\[
\sum_{n,m} |\langle f, g_{nm} \rangle|^2 \leq B_g \|f\|^2 , \quad f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) , \tag{1.1.4}
\]
and any \( B_g < \infty \) such that (1.1.4) holds is called a frame upper bound.

When the system (1.1.1) has a finite frame upper bound \( B_g \), one can define an analysis operator \( T_g \) and a synthesis operator \( T_g^* \) by
\[
T_g : f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \rightarrow T_g f = (\langle f, g_{nm} \rangle)_{n,m} , \tag{1.1.5}
\]
and
\[
T_g^* : \alpha \in l^2(\mathbb{Z} \times I) \rightarrow T_g^* \alpha = \sum_{n,m} \alpha_{nm} g_{nm} \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) , \tag{1.1.6}
\]
respectively. These \( T_g \) and \( T_g^* \) are bounded linear operators with operator norm \( \leq B_g^{1/2} \). Observe that \( T_g \) and \( T_g^* \) as defined by (1.1.5) and (1.1.6) are indeed adjoint operators with respect to the inner product \( \langle \cdot, \cdot \rangle \). When the system (1.1.2) has a finite frame upper bound as well, the duality condition (1.1.3) can be expressed as
\[
T_g^* T_g = I , \tag{1.1.7}
\]
where \( I \) denotes the identity operator of \( L^2(\mathbb{R}) \) or \( l^2(\mathbb{Z}) \).

When the system (1.1.1) has a finite frame upper bound \( B_g \), the frame operator \( S_g \) is defined by \( S_g = T_g^* T_g \). Explicitly,
\[
S_g : f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \rightarrow S_g f = \sum_{n,m} \langle f, g_{nm} \rangle g_{nm} \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) , \tag{1.1.8}
\]
and there holds
\[
S_g \leq B_g I . \tag{1.1.9}
\]
When there is a \( A_g > 0 \) such that
\[
\sum_{n,m} |\langle f, g_{nm} \rangle|^2 \geq A_g \|f\|^2 , \quad f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) , \tag{1.1.10}
\]
or, equivalently, \( S_g \) is invertible with
\[
S_g \geq A_g I , \tag{1.1.11}
\]
we say that the system (1.1.1) has a positive lower frame bound, and any \( A_g > 0 \) such that (1.1.10) holds is called a lower frame bound. In the case that the system (1.1.1) has both a finite frame upper bound and a positive frame lower bound, we say that (1.1.1) is a frame. Then a dual system is given by
\[
\gamma_{nm} = (S_g^{-1} g_{mn})(\cdot - n\Delta), \quad (n, m) \in \mathbb{Z} \times I , \tag{1.1.12}
\]
and this system is also a frame. It is essential to note here that the frame operator \( S_g \) commutes with all relevant time-shift operators. More generally, when the systems (1.1.1) and (1.1.2) have finite frame upper bounds and (1.1.3) holds for all \( f \), both systems are a frame. The dual system (1.1.12) is special in the following sense: for any \( f \in L^2(\mathbb{R}) \) or \( l^2(\mathbb{Z}) \) and any \( \alpha \in l^2(\mathbb{Z} \times I) \) with
\[
f = \sum_{n,m} \alpha_{nm} g_{nm} \tag{1.1.13}
\]
there holds
\[
\sum_{n,m} |\langle f, \gamma_{nm} \rangle|^2 \leq \sum_{n,m} |\alpha_{nm}|^2 \tag{1.1.14}
\]
with equality if and only if \( \langle f, \gamma_{nm} \rangle = \alpha_{nm} \) for all \( (n, m) \in \mathbb{Z} \times I \). One checks, furthermore, easily that \( S_g S_{\gamma_1} = I \), so that \( S_{\gamma_1} \) is the inverse frame operator. We refer to this dual system as the minimal dual system. For generalities about frames and shift-invariant systems we refer to [Dau90], Sec. 3.2, [Dau92] Sec. II, and [RS95b], Sec. 1.3.

A particular example of a shift-invariant system arises when we take
\[
g_m(t) = e^{2\pi i m t} g(t) , \quad t \in \mathbb{R}, \quad m \in \mathbb{Z} , \tag{1.1.15}
\]
with \( b > 0 \) and \( g \in L^2(\mathbb{R}) \) (time-continuous case), or
\[
g_m(j) = e^{2\pi i m j/M} g(j) , \quad j \in \mathbb{Z}, \quad m = 0, \ldots, M - 1 , \tag{1.1.16}
\]
with \( M \in \mathbb{N} \) and \( g \in l^2(\mathbb{Z}) \) (time-discrete case). It is customary here to ignore the phase factors in \( g_{nm}, \gamma_{nm} \) when studying duality questions, since these vanish anyway at the right-hand side of (1.1.3). Hence one has for the time-continuous case
\[
g_m(t) = e^{2\pi i m t} g(t) \quad \text{rather than} \quad e^{2\pi i m(t-na)} g(t-na) \tag{1.1.17}
\]
for \((n, m) \in \mathbb{Z} \times \mathbb{Z}\), etc. We refer to the systems so obtained as \((\text{time-continuous or time-discrete})\) Weyl-Heisenberg systems, and the \(g\) in \((1.1.15)\) or \((1.1.16)\) is sometimes called a prototype function or sequence. Since the frame operator \(S_g\) commutes now with all relevant time-shift and frequency-shift operators, it turns out that when \(g_{nm}, (n, m) \in \mathbb{Z} \times \mathbb{I}\), is a Weyl-Heisenberg frame, then so is the minimal dual system \(\gamma_{nm}, (n, m) \in \mathbb{Z} \times \mathbb{I}\).

In the next sections we work out the condition of duality for two shift-invariant systems, and in particular two Weyl-Heisenberg systems, in various domains. Specifically, we present for a shift-invariant system \(g_{nm}, (n, m) \in \mathbb{Z} \times \mathbb{I}\), a necessary and sufficient condition, in terms of the Fourier transforms of the \(g_n\), to be a frame. Also, we present a necessary and sufficient condition for two shift-invariant systems \(g_{nm}, (n, m) \in \mathbb{Z} \times \mathbb{I}\), and \(\gamma_{nm}, (n, m) \in \mathbb{Z} \times \mathbb{I}\), both having a finite frame upper bound, to be dual. Furthermore, we give a representation result for frame operators in the Fourier domain; we present a method to compute the minimal dual system and we characterize it in the Fourier domain. By specialization to Weyl-Heisenberg systems, we obtain formulations of the conditions of having a finite frame upper bound, duality, etc., both in the frequency domain and in the time domain. A third domain where these conditions and computation methods are worked out is the time-frequency domain. This consists of a detailed elaboration of the celebrated Wexler–Raz condition \([WR90]\) of biorthogonality of two Weyl-Heisenberg systems. Finally, the conditions and computation methods for Weyl-Heisenberg systems are worked out in a fourth domain, viz. the Zak transform domain; for time-continuous Weyl-Heisenberg systems it is required here that the product \(\Delta \cdot b\) of the two shift parameters, see \((1.1.1)\) and \((1.1.15)\), is rational.

There are some good reasons for working out the duality condition in various domains. First of all there is the issue of computational advantages in calculating (minimal) dual systems which can be easier in one domain than in the other. For instance, as we shall see in Subsec. 1.4.3, the frame operator \(S_g\) (time-continuous case) has the representation

\[
S_g = \frac{1}{ab} \sum_{k,l} \langle g, U_{kl} g \rangle U_{kl}
\]

in the time-frequency domain, where \(U_{kl}\) is the time-frequency shift operator

\[
(U_{kl} f)(t) = e^{2\pi i kl/a} f(t - k/b), \ t \in \mathbb{R},
\]

declared for \(f \in L^2(\mathbb{R})\). It can be expected, certainly when \(ab \ll 1\), that the representation \((1.1.18)\) is more efficient than the "direct" representation \((1.1.17)\) with \(g_{nm}\) given by \((1.1.17)\). Hence computation and inversion of the frame operator is more easily done by using \((1.1.18)\) than by using \((1.1.17)\). Also, the advantages can be of a more algebraic nature, in the sense that the matrices and operators one has to calculate with are sparser or structured.

A second reason to consider duality in different domains arises when one wants to use \((1.1.3)\) as an efficient way to represent signals \(f\) by means of the expansion coefficient \((f, \gamma_{nm})\). In Subsec. 1.6.2 we shall present a number of constraints, as they may arise in certain data-storage and -transmission applications, to which such a representation method may be subjected. (Note that for applications of this type, it is almost pointless to want to calculate dual systems at high speed: it is the signal processing involved in using these systems that should be fast.) Now the constraints just mentioned are of several types. There are hard ones, such as the condition of duality, and there are soft ones, such as the condition of good frequency discrimination or smoothness of the constituent filter impulse responses. Also, the constraints may have been formulated in various domains (notably, but not exclusively, in the time domain and in the frequency domain). A possible strategy to design filter impulse responses satisfying some or many of the constraints could be to introduce a cost functional that incorporates soft constraints, and to minimize this functional under the condition that hard constraints are satisfied. Since one may have to switch here iteratively from one domain to another, it would be very convenient when the hard constraints were formulated in any of the domains in which the soft constraints are formulated. As to the duality constraint we have been successful in this respect.

Most of the results presented here are proved somewhere in the literature, quite often in a form that is accessible for most of the readers. This is perhaps not so for some of the results for the time-discrete case. However, usually the proofs of the latter results consist of slight adaptations of the proofs for the corresponding time-continuous results. The reason for nevertheless including the results (but not their proofs) for the time-discrete case is that the formulation of these results does not always follow straightforwardly from their time-continuous counterparts. For these reasons most of the proofs are omitted (with appropriate references, however, whenever possible). An exception has been made for the material presented in Sec. 1.2 on shift-invariant systems, that was also covered for the most part by Ron and Shen in \([RS95b]\), and for the proof of the fundamental identity in Sec. refs:4 on which the Wexler-Raz approach is based.

### 1.2 Time-continuous shift-invariant systems

In this section we consider time-continuous shift-invariant systems, i.e. we take \(\mathbb{I} = \mathbb{Z}, \Delta = a > 0\), and we have \(g_{nm}, \gamma_{nm} \in L^2(\mathbb{R})\) in \((1.1.1), (1.1.2)\). We shall present, in the frequency domain, equivalent conditions for a sys-
tem $g_{nm}, (n, m) \in \mathbb{Z}^2$, to have a finite frame upper bound and to be a frame, an equivalent condition for two systems $g_{nm}, (n, m) \in \mathbb{Z}^2$, and $\gamma_{nm}, (n, m) \in \mathbb{Z}^2$, to be dual, a characterization of the minimal dual system, and a resulting algorithm to compute the minimal dual system. Furthermore, we give a representation result for the frame operator in the frequency domain, and we present a link with polyphase operators. Some of the results in this section can be found in the recent paper [RS95b] by Ron and Shen. However, their presentation is rather different from ours, and we have not been able to find the results on frame operator representation and on computation of minimal dual systems, so we have chosen to give all proofs here.

We first need some preparation. We consider $L^2(\mathbb{R})$ with the inner product norm $\|f\| = \langle f, f \rangle^{1/2}$, where

$$(f, h) = \int_{-\infty}^{\infty} f(t) h^*(t) \, dt \ , \ f, h \in L^2(\mathbb{R}) \quad (1.2.1)$$

Also, we let $T_x$ be the translation operator

$$(T_x f)(t) = f(t + x) \ , \ t \in \mathbb{R}, \ f \in L^2(\mathbb{R}) \quad (1.2.2)$$

where $x \in \mathbb{R}$ (since we use $T_x$ only in Prop. 2.1 below, there is no confusion possible with the analysis operator $T_g$ of (1.1.5)). Finally, we denote for $h \in L^2(\mathbb{R})$ by $\hat{h} = \mathcal{F} h$ the Fourier transform

$$\hat{h}(\nu) = (\mathcal{F} h)(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i \nu t} h(t) \, dt \ , \ \text{a.e. } \nu \in \mathbb{R} \quad (1.2.3)$$

**Proposition 1.2.1** Assume that the systems $g_{nm} = g_m(-na), (n, m) \in \mathbb{Z}^2$, and $\gamma_{nm} = \gamma_m(-na), (n, m) \in \mathbb{Z}^2$, have finite frame upper bounds $B_g, B_\gamma$ (the systems do not need to be related by duality). Then

$$\sum_m |\tilde{g}_m(\nu)|^2 \leq a B_g \ , \ \sum_m |\tilde{\gamma}_m(\nu)|^2 \leq a B_\gamma \ , \ \text{a.e. } \nu \in \mathbb{R} \quad (1.2.4)$$

When $f, h \in L^2(\mathbb{R})$, the function

$$\rho(f, h)(x) = \sum_{n,m} (T_x f, g_{nm}) (\gamma_{nm}, T_x h) \quad (1.2.5)$$

is continuous and periodic in $x$ with period $a$, and has the Fourier series

$$\rho(f, h)(x) \sim \sum_k c_k e^{-2\pi ikx/a} \quad (1.2.6)$$

$$c_k = \frac{1}{a} \int_{-\infty}^{\infty} \hat{f}(\nu) \hat{h}^*(\nu + k/a) \sum_m \tilde{g}_m^*(\nu) \tilde{\gamma}_m(\nu + k/a) \, d\nu \ , \ k \in \mathbb{Z} \quad (1.2.7)$$

**Proof:** First assume that $f, h \in S$ (Schwartz space of smooth and rapidly decaying functions). By absolute and bounded convergence of the series in (1.2.5) we have

$$c_k = \frac{1}{a} \int_{-\infty}^{\infty} e^{2\pi i kx/a} \sum_{n,m} \langle f, g_m(-x-na) \rangle \langle h, \gamma_m(-x-na) \rangle^* \, dx =$$

$$= \frac{1}{a} \sum_{m} \int_{-\infty}^{\infty} e^{2\pi i kx/a} \langle f, g_m(-x) \rangle \langle h, \gamma_m(-x) \rangle^* \, dx \quad (1.2.8)$$

Now for $m \in \mathbb{Z}, \mu \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} e^{2\pi i \mu x} \langle f, g_m(-x) \rangle \langle h, \gamma_m(-x) \rangle^* \, dx =$$

$$= \int_{-\infty}^{\infty} e^{2\pi i \mu x} \left( \int_{-\infty}^{\infty} e^{2\pi i \nu x} \hat{f}(\nu) \tilde{g}_m^*(\nu) \, d\nu \right) \langle h, \gamma_m(\nu + \mu) \rangle^* \, d\nu =$$

$$= \int_{-\infty}^{\infty} \hat{f}(\nu) \tilde{g}_m^*(\nu) \left( \hat{h}(\nu + \mu) \gamma_m^*(\nu + \mu) \right) \, d\nu ,$$

where we have applied Parseval’s theorem to the functions $\hat{f} \cdot \tilde{g}_m^* \in L^2(\mathbb{R})$ and $\hat{h}(\cdot + \mu) \gamma_m^*(\cdot + \mu) \in L^2(\mathbb{R})$. Accordingly,

$$c_k = \frac{1}{a} \sum_{m} \int_{-\infty}^{\infty} \hat{f}(\nu) \hat{h}^*(\nu + k/a) \tilde{g}_m^*(\nu) \gamma_m(\nu + k/a) \, d\nu \quad (1.2.10)$$

Now take $f = h, g_m = \gamma_m, k = 0$ in (1.2.10). Then

$$\frac{1}{a} \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 \sum_m |\tilde{g}_m(\nu)|^2 \, d\nu = a_0 =$$

$$= \frac{1}{a} \int_{0}^{a} \sum_{n,m} |(T_x f, g_{nm})|^2 \, dx \leq B_g \|f\|^2 = B_g \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 \, d\nu \quad (1.2.11)$$

From this the first (and, similarly, the second) inequality in (1.2.4) follows. Next, we interchange the sum and the integral in (1.2.10), and (1.2.7) follows.
For general \( f, h \in L^2(\mathbb{R}) \) we get (1.2.7) easily from a density argument (observe that the right-hand side integral in (1.2.7) converges absolutely). Finally, the continuity of \( \rho(f, h) \) can be shown by using the frame upper bound conditions on the two systems \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), and \( \gamma_{nm}, (n, m) \in \mathbb{Z}^2 \), together with the fact that \( \|T_x f - f\| \to 0 \) as \( x \to 0 \) when \( f \in L^2(\mathbb{R}) \).

**Theorem 1.2.2** Assume that the systems \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), and \( \gamma_{nm}, (n, m) \in \mathbb{Z}^2 \), have finite frame upper bounds. Then the systems are dual in the sense that

\[
(f, h) = \sum_{n, m} \langle f, g_{nm} \rangle \langle \gamma_{nm}, h \rangle, \quad f, h \in L^2(\mathbb{R}),
\]

(1.2.12)

if and only if

\[
\varphi_k(\nu) := \sum_m \hat{g}_m(\nu - k/a) \hat{\gamma}_m(\nu) = a \delta_{k0}, \quad \text{a.e. } \nu \in \mathbb{R}, \quad k \in \mathbb{Z}.
\]

(1.2.13)

**Proof:** Assume that the two systems are dual, and let \( f, h \in L^2(\mathbb{R}) \). Then, see Prop. 2.1,

\[
\rho(f, h)(x) = \langle f, h \rangle, \quad x \in \mathbb{R}.
\]

(1.2.14)

Hence by (1.2.7)

\[
\int_{-\infty}^{\infty} \hat{f}(\nu - k/a) \hat{h}^*(\nu) \varphi_k(\nu) d\nu = a \delta_{k0} \int_{-\infty}^{\infty} \hat{f}(\nu) \hat{h}^*(\nu) d\nu, \quad k \in \mathbb{Z},
\]

(1.2.15)

and (1.2.13) easily follows.

Conversely, assume that (1.2.13) holds. Then by (1.2.6)–(1.2.7) for \( f, h \in L^2(\mathbb{R}) \)

\[
\rho(f, h)(x) \sim \langle f, h \rangle.
\]

(1.2.16)

Since both sides of (1.2.16) are continuous functions of \( x \), they agree for all \( x \in \mathbb{R} \). Taking \( x = 0 \) we get (1.2.12), and the proof is complete.

To proceed we need some preparation and a definition. We let

\[
H_\beta(\nu) := (\hat{g}_m(\nu - k/a))_{k \in \mathbb{Z}, m \in \mathbb{Z}}, \quad \text{a.e. } \nu \in \mathbb{R}.
\]

(1.2.17)

**Proposition 1.2.3** Assume that the system \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), has a finite frame upper bound \( B_\beta \). Then \( H_\beta(\nu) \) of (1.2.17) defines for a.e. \( \nu \in \mathbb{R} \) a bounded linear operator of \( L^2(\mathbb{Z}) \) with operator norm \( \leq (a B_\beta)^{1/2} \). Explicitly, we have for a.e. \( \nu \in \mathbb{R} \)

\[
\sum_k \left| \sum_m \hat{g}_m(\nu - k/a) \beta_m \right|^2 \leq a B_\beta \|\beta\|^2, \quad \beta \in L^2(\mathbb{Z}),
\]

(1.2.18)

where \( \|\beta\| = (\sum_m |\beta_m|^2)^{1/2} \) is the norm of \( \beta \in L^2(\mathbb{Z}) \).

**Proof:** Let \( \alpha_{nm} \neq 0 \) for only finitely many \( n, m \), and let

\[
\alpha_{nm}(\nu) = \sum_n \alpha_{nm} e^{-2\pi ina \nu}, \quad \nu \in \mathbb{R}.
\]

(1.2.19)

Also let \( J \) be an interval of length \( 1/a \). Then we have

\[
\int_J \sum_k \sum_m \alpha_{nm}(\nu) \hat{g}_m(\nu - k/a) d\nu = \int_{-\infty}^{\infty} \left| \sum_m \alpha_{nm}(\nu) \hat{g}_m(\nu) \right|^2 d\nu
\]

\[
= \int_{-\infty}^{\infty} \left| \sum_n \alpha_{nm} e^{-2\pi ina \nu} \hat{g}_m(\nu) \right|^2 d\nu = \left\| \sum_n \alpha_{nm} g_{nm} \right\|^2,
\]

(1.2.20)

where we have used 1/a-periodicity of the \( \alpha_{nm} \) and Parseval’s theorem. Now, see below (1.1.6), the far right-hand side of (1.2.20) is bounded by \( B_\beta \|\alpha\|^2 \), and since

\[
\|\alpha\|^2 = \sum_{n, m} |\alpha_{nm}|^2 = a \int_J \sum_m |\alpha_{nm}(\nu)|^2 d\nu,
\]

(1.2.21)

we get

\[
\int_J \sum_k \sum_m \alpha_{nm}(\nu) \hat{g}_m(\nu - k/a) d\nu \leq a B_\beta \int_J \sum_m |\alpha_{nm}(\nu)|^2 d\nu.
\]

(1.2.22)

Next fix \( \beta \in L^2(\mathbb{Z}) \) with \( \beta_m \neq 0 \) for only finitely many \( m \in \mathbb{Z} \), and choose in (1.2.19)

\[
\alpha_{nm}(\nu) = \beta_m \varphi(\nu); \quad \varphi(\nu) = \sum_n \varphi_n e^{-2\pi ina \nu},
\]

(1.2.23)

with \( \varphi_n \neq 0 \) for only finitely many \( n \in \mathbb{Z} \). Then (1.2.22) gives

\[
\int |\varphi(\nu)|^2 \sum_k \sum_m \hat{g}_m(\nu - k/a) \beta_m^2 d\nu \leq a B_\beta \|\beta\|^2 \int |\varphi(\nu)|^2 d\nu.
\]

(1.2.24)
By varying $\phi$ over all allowed $\frac{1}{a}$-periodic functions, we see that
\[
\sum_k \left| \sum_m \hat{g}_m(\nu - k/a) \beta_m \right|^2 \leq a B_g \|\beta\|^2, \text{a.e. } \nu \in J, \tag{1.2.25}
\]
where the null set involved in (1.2.25) may depend on $\beta$.

Now let $\mathcal{V}$ be a dense, countable set in $L^2(\mathbb{Z})$ of $\beta$'s with $\beta_m \neq 0$ for only finitely many $m \in \mathbb{Z}$, and let $N_1 \subset J$ be a null set outside of which
\[
\sum_k \left| \sum_m \hat{g}_m(\nu - k/a) \beta_m \right|^2 \leq a B_g \|\beta\|^2, \quad \beta \in \mathcal{V}. \tag{1.2.26}
\]
Also, let $N_2 \subset J$ be a null set outside of which
\[
\sum_m |\hat{g}_m(\nu - k/a)|^2 \leq a B_g, \quad k \in \mathbb{Z}, \tag{1.2.27}
\]
see Prop. 2.1. Then take $\beta \in L^2(\mathbb{Z})$, and let $\beta^{(M)} \in \mathcal{V}$ such that $\|\beta^{(M)} - \beta\| \to 0$ as $M \to \infty$. When $\nu \notin N_1 \cup N_2$, we have by Fatou's lemma
\[
\sum_k \left| \sum_m \hat{g}_m(\nu - k/a) \beta_m^{(M)} \right|^2 = \sum_k \liminf_{M \to \infty} \left| \sum_m \hat{g}_m(\nu - k/a) \beta_m^{(M)} \right|^2 \\
\leq \liminf_{M \to \infty} \sum_k \left| \sum_m \hat{g}_m(\nu - k/a) \beta_m^{(M)} \right|^2 \\
\leq a B_g \sum_m |\hat{g}_m(\nu)|^2 \leq a B_g^2. \tag{1.2.28}
\]
We have shown now that for any interval $J$ of length $1/a$ there is a null set $N \subset N_1 \cup N_2$ such that
\[
\sum_k \left| \sum_m \hat{g}_m(\nu - k/a) \beta_m \right|^2 \leq a B_g \|\beta\|^2, \quad \beta \in L^2(\mathbb{Z}), \quad \nu \in J \setminus N. \tag{1.2.29}
\]
This completes the proof.

We are now ready to give a representation result in the frequency domain of the frame operator
\[
S_gf = \sum_{n,m} \langle f, g_{nm} \rangle g_{nm}, \quad f \in L^2(\mathbb{R}). \tag{1.2.30}
\]
For the case of Weyl-Heisenberg frames the formulas (1.2.31)-(1.2.32) below are also known as the Walnut representation of the frame operator, see [Wal92], Prop. 2.4.

**Theorem 1.2.4** Assume that the system $g_{nm}, \quad (n,m) \in \mathbb{Z}^2$, has a finite frame upper bound $B_g$, and let $f \in L^2(\mathbb{R})$. Then we have
\[
\overline{S_g f}(\nu) = \frac{1}{a} \sum_k d_k(\nu) \hat{f}(\nu - k/a) \tag{1.2.31}
\]
with absolute convergence for a.e. $\nu \in \mathbb{R}$, where, see (1.2.17),
\[
d_k(\nu) = \langle H_g(\nu) H_g^*(\nu) \rangle_{\mathcal{C}_k} = \sum_m \hat{g}_m(\nu) \hat{g}_m^*(\nu - k/a), \quad k \in \mathbb{Z}. \tag{1.2.32}
\]

**Proof:** We have by Props. 2.1 and 2.3
\[
\sum_k |d_k(\nu)|^2 \leq \sum_k \left| \sum_m \hat{g}_m(\nu) \hat{g}_m^*(\nu - k/a) \right|^2 \leq \sum_k \sum_m |\hat{g}_m(\nu)|^2 \leq (a B_g)^2, \tag{1.2.33}
\]
for a.e. $\nu \in \mathbb{R}$. Since $\hat{f} \in L^2(\mathbb{R})$, so that $\sum_k |\hat{f}(\nu - k/a)|^2 < \infty$ for a.e. $\nu \in \mathbb{R}$, we see that the right-hand side of (1.2.31) is a.e. well-defined as an absolutely convergent series. Also, by the Cauchy-Schwarz inequality, the right-hand side of (1.2.31) represents an $L^2_{\text{loc}}(\mathbb{R})$-function.

Now let $h \in \mathcal{S}$. We shall show that
\[
\langle S_g f, h \rangle = \frac{1}{a} \int_{-\infty}^{\infty} \left( \sum_k d_k(\nu) \hat{f}(\nu - k/a) \right) \hat{h}(\nu) d\nu. \tag{1.2.34}
\]
From this and the above, the result follows by density of $\mathcal{S}$ in $L^2(\mathbb{R})$ and Parseval’s theorem.

To show (1.2.34) we observe that by (1.2.33) and the Cauchy-Schwarz inequality
\[
\sum_k \int_{-\infty}^{\infty} |d_k(\nu)||\hat{f}(\nu - k/a)||\hat{h}(\nu)| d\nu \leq (a B_g)^2 \int_{-\infty}^{\infty} \left( \sum_k |\hat{f}(\nu - k/a)|^2 \right)^{\frac{1}{2}} |\hat{h}(\nu)| d\nu. \tag{1.2.35}
\]
The right-hand side of (1.2.35) is easily seen to be finite, whence the function $\rho(f, h)$ of Prop. 2.1 with $\gamma_m = g_m, m \in \mathbb{Z}$, has an absolutely convergent
Fourier series. Therefore
\[
\langle S f, h \rangle = \rho(f, h)(0) = \sum_k c_k = \sum_k \frac{1}{a} \int_{-\infty}^{\infty} \hat{f}(\nu) \hat{h}(\nu + k/a) \sum_m \hat{g}(\nu + k/a) d\nu.
\]
By (1.2.35) the series and the integral in the last member of (1.2.36) may be interchanged, and we arrive at (1.2.34). This completes the proof.

We next present a basic result, also to be found in [RS95b], on frame bounds in terms of the operators \(H_{\nu}(\nu)\) of (1.2.17).

**Theorem 1.2.5** Let \(g_n \in L^2(\mathbb{R}), m \in \mathbb{Z}\), and let \(A \geq 0, B < \infty\). Then
\[
A \|f\|^2 = \sum_{n,m} |(f, g_{nm})|^2 \leq B \|f\|^2, f \in L^2(\mathbb{R})
\] (1.2.37)
\[
\Leftrightarrow a A I \leq H_{\nu}(\nu) H_{\nu}^*(\nu) \leq A B I, \text{ a.e. } \nu \in \mathbb{R},
\]
where \(I\) denotes the identity operator of \(L^2(\mathbb{Z})\).

**Proof:** When at least one of the members of (1.2.37) holds, we have that \(H_{\nu}(\nu) H_{\nu}^*(\nu) \leq A B I\) by Prop. 2.3 since \(H_{\nu}(\nu)\) and \(H_{\nu}^*(\nu)\) have the same operator norm. Next let \(f \in S\) with \(\hat{f}\) compactly supported, and let \(m \in \mathbb{Z}\). Then holds the \(L^2(\mathbb{R}^*)\)-convergent Fourier expansion
\[
\sum_k \hat{g}(\nu - k/a) \hat{f}^*(\nu - k/a) = \sum_n c_n e^{2\pi in\nu}(1.2.38)
\]
with
\[
c_n = \langle g_n, f \rangle, n \in \mathbb{Z}.
\] (1.2.39)
Thus, when \(J\) is an interval of length \(a^{-1}\),
\[
\int_{\nu} \sum_k |\hat{g}(\nu - k/a) \hat{f}^*(\nu - k/a)|^2 d\nu = a \sum_n |(f, g_{nm})|^2.
\] (1.2.40)
Denote for \(\nu \in \mathbb{R}\)
\[
\hat{f}(\nu) = \left(\hat{f}(\nu - k/a)\right)_{k \in \mathbb{Z}}.
\] (1.2.41)
Observe that for any interval \(J\) of length \(a^{-1}\) we have
\[
\|\hat{f}\|^2 = \int_J \|\hat{f}(\nu)\|^2 d\nu.
\] (1.2.42)
Then, when at least one of the members of (1.2.37) holds, we see from (1.2.40) that for any interval \(J\) of length \(a^{-1}\)
\[
\int_J \|H_{\nu}^*(\nu) \hat{f}(\nu)\|^2 d\nu = a \sum_n \int_{\nu} \|\hat{g}(\nu - k/a) \hat{f}^*(\nu - k/a)\|^2 d\nu
\] (1.2.43)
\[
= a \sum_{n,m} \|f, g_{nm}\|^2.
\]
\[
\Rightarrow \text{ By what has been said above we only have to show that } \|H_{\nu}^*(\nu) \beta\|^2 \geq A \|\beta\|^2 \text{ for } \beta \in L^2(\mathbb{Z}). \text{ Let } \hat{f} \in S \text{ be supported by an interval } J \text{ of length } a^{-1}, \text{ and let } \hat{\beta} \in L^2(\mathbb{Z}) \text{ with } \hat{\beta}_k \neq 0 \text{ for only finitely many } k \in \mathbb{Z}. \text{ For } \nu \in \mathbb{R} \text{ define } \hat{f}(\nu) := \beta_k \hat{f}(\nu + k/a), \text{ where } k \in \mathbb{Z} \text{ is such that } \nu + k/a \in J. \text{ This } f \text{ satisfies}
\]
\[
\hat{f}(\nu) = \hat{\phi}(\nu) \beta, \nu \in J.
\] (1.2.44)
Therefore we get from (1.2.43), the first member of (1.2.37) and (1.2.42) that
\[
\int_J 
\tau(\nu)^2 \|H_{\nu}^*(\nu) \beta\|^2 d\nu = a \sum_{n,m} |(f, g_{nm})|^2 \geq a A \|f\|^2
\] (1.2.45)
\[
= a A \int_J \|\hat{f}(\nu)\|^2 d\nu = a A \|\hat{\beta}\|^2 \int_J |\hat{\phi}(\nu)|^2 d\nu.
\]
By varying \(\hat{\phi}\) over all elements of \(S\) supported by \(J\) we get
\[
\|H_{\nu}^*(\nu) \beta\|^2 \geq a A \|\hat{\beta}\|^2, \text{ a.e. } \nu \in J,
\] (1.2.46)
where the null set involved in (1.2.46) may depend on \(\hat{\beta}\).
Now let \(V\) be a countable dense set in \(L^2(\mathbb{Z})\) of \(\beta\)'s such that \(\beta_k \neq 0\) for only finitely many \(k \in \mathbb{Z}\), and let \(N_1 \subset J\) be a null set such that
\[
\|H_{\nu}^*(\nu) \beta\|^2 \geq a A \|\hat{\beta}\|^2, \beta \in V, \nu \in J \setminus N_1.
\] (1.2.47)
Also, let \(N_2 \subset J\) be a null set such that
\[
\|H_{\nu}^*(\nu) \beta\|^2 \leq a B \|\hat{\beta}\|^2, \beta \in L^2(\mathbb{Z}), \nu \in J \setminus N_2.
\] (1.2.48)
When now \( \nu \in J \setminus (N_1 \cup N_2) \) and \( \beta \in L^2(\mathbb{Z}) \), we choose \( \beta^{(M)} \in V \) with \( \|\beta^{(M)} - \beta\| \to 0 \) as \( M \to \infty \), and we conclude from (1.2.47) and (1.2.48) that
\[
\|H_\nu^* (\nu) \beta\|^2 = \lim_{M \to \infty} \|H_\nu^* (\nu) \beta^{(M)}\|^2 \geq a A \lim_{M \to \infty} \|\beta^{(M)}\|^2 = a A \|\beta\|^2 .
\]
Then the proof of \( \Rightarrow \) is easily completed.

\( \Leftarrow \). Let \( f \in S \) with \( f \) boundedly supported and let \( J \) be an interval of length \( a^{-1} \). Then (1.2.42) and (1.2.43) show that the quantity \( a \sum_{n,m} |(f, g_{nm})|^2 \) lies between \( a \|f\|^2 \) and \( a B \|f\|^2 \). From this we get the first member of (1.2.37) and the proof is complete.

We next present a result on the computation of minimal dual systems and a characterization of these systems in terms of generalized inverses.

**Theorem 1.2.6** Assume that the system \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), is a frame, and denote by \( \xi(\nu) \in L^2(\mathbb{Z}) \) for a.e. \( \nu \in \mathbb{R} \) the least-norm solution \( \xi = (c_m)_{m \in \mathbb{Z}} \) of the linear system
\[
\sum_m g_m(\nu - k/a) c_m = a \delta_{k0}, \quad k \in \mathbb{Z} .
\]
Then there holds
\[
^o \tilde{\gamma}_m(\nu) = c_m^*(\nu), \quad m \in \mathbb{Z}, \text{ a.e. } \nu \in \mathbb{R} .
\]

**Proof:** When \( \xi = (\delta_{k0})_{k \in \mathbb{Z}} \), the least-norm solution
\[
\xi(\nu) = a H_\nu^*(\nu) (H_\nu(\nu) H_\nu^*(\nu))^{-1} \xi
\]
of \( H_\nu(\nu) \xi = a \xi \) is explicitly given by
\[
c_m(\nu) = a \sum_k \left( (H_\nu(\nu) H_\nu^*(\nu))^{-1} \right)_{k0} g_m(\nu - k/a),
\quad m \in \mathbb{Z}, \text{ a.e. } \nu \in \mathbb{R} .
\]

On the other hand by Theorem 2.4
\[
\tilde{g}_m(\nu) = S_g \circ \gamma_m(\nu) = \frac{1}{a} \sum_k (H_\nu(\nu) H_\nu^*(\nu))_{ok}^o \tilde{\gamma}_m(\nu - k/a),
\quad m \in \mathbb{Z}, \text{ a.e. } \nu \in \mathbb{R} .
\]
Now using the fact that for all \( I, k \in \mathbb{Z} \)
\[
(H_\nu(\nu - l/a) H_\nu^*(\nu - l/a))_{ok} = (H_\nu(\nu) H_\nu^*(\nu))_{l,l+k}, \text{ a.e. } \nu \in \mathbb{R} ,
\]
we get for all \( l \in \mathbb{Z} \)
\[
\hat{g}_m(\nu - l/a) = \frac{1}{a} \sum_k (H_\nu(\nu) H_\nu^*(\nu))_{ok}^o \tilde{\gamma}_m(\nu - k/a),
\quad m \in \mathbb{Z}, \text{ a.e. } \nu \in \mathbb{R} .
\]
Letting for \( m \in \mathbb{Z} \)
\[
\hat{g}_m(\nu) = (\hat{g}_m(\nu - l/a))_{l \in \mathbb{Z}}, \quad \hat{\gamma}_m(\nu) = (\hat{\gamma}_m(\nu - k/a))_{k \in \mathbb{Z}},
\quad \text{a.e. } \nu \in \mathbb{R} ,
\]
we thus have
\[
\hat{g}_m(\nu) = \frac{1}{a} H_\nu(\nu) H_\nu^*(\nu) \hat{\gamma}_m(\nu), \quad m \in \mathbb{Z}, \text{ a.e. } \nu \in \mathbb{R} .
\]
That is
\[
^o \hat{\gamma}_m(\nu) = a (H_\nu(\nu) H_\nu^*(\nu))^{-1} \hat{g}_m(\nu),
\quad m \in \mathbb{Z}, \text{ a.e. } \nu \in \mathbb{R} .
\]
When we write out (1.2.59) for the \( 0 \)-th coordinate, we get
\[
^o \hat{\gamma}_m(\nu) = a \left( \sum_k (H_\nu(\nu) H_\nu^*(\nu))^{-1} \right)_{0k} \hat{g}_m(\nu - k/a),
\quad m \in \mathbb{Z}, \text{ a.e. } \nu \in \mathbb{R} .
\]
The right-hand side of (1.2.60) equals \( c_m^*(\nu) \) by (1.2.52) (observe that \( H_\nu(\nu) H_\nu^*(\nu) \) is Hermitian), and the proof is complete.

**Theorem 1.2.7** Assume that the systems \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), and \( \gamma_{nm}, (n, m) \in \mathbb{Z}^2 \), have finite frame upper bounds. Then they are dual if and only if
\[
H_\nu(\nu) H_\nu^*(\nu) = a I, \text{ a.e. } \nu \in \mathbb{R} .
\]
Moreover,
\[
H_\nu^*(\nu) = a H_\nu^*(\nu) (H_\nu(\nu) H_\nu^*(\nu))^{-1}, \text{ a.e. } \nu \in \mathbb{R} ,
\]
so that
\[
\frac{1}{a} H_\nu^*(\nu) H_\nu^*(\nu) = (\frac{1}{a} H_\nu(\nu) H_\nu^*(\nu))^{-1}, \text{ a.e. } \nu \in \mathbb{R} .
\]
Proof: We can write (1.2.61) as
\[ \sum_m \hat{g}_m(\nu - k/a) \hat{\gamma}_m(\nu - l/a) = a \delta_{kl} , \quad k, l \in \mathbb{Z} , \quad \text{a.e. } \nu \in \mathbb{R} . \]
(1.2.64)

This is just (1.2.13) applied with \( \nu - l/a \) instead of \( \nu \).

To show (1.2.62) we just note that (1.2.59) implies that
\[ H_{s\nu}(\nu) = a \left( H_g(\nu) H_g^*(\nu) \right)^{-1} H_g(\nu) , \quad \text{a.e. } \nu \in \mathbb{R} , \]
(1.2.65)
and (1.2.62) follows upon conjugation. Finally, (1.2.63) is a direct consequence of (1.2.62), and the proof is complete.

As a consequence of (1.2.63) and Theorem 2.4 we see that the inverse frame operator \( S_{s\nu}^{-1} = S_{s\nu} \) corresponding to the minimal dual system has the representation
\[ S_{s\nu} f(\nu) = \sum_k \left( \frac{1}{a} H_g(\nu) H_g^*(\nu) \right)^{-1} f(\nu - k/a) , \quad \text{a.e. } \nu \in \mathbb{R} , \]
(1.2.66)
when \( f \in L^2(\mathbb{R}) \).

We conclude this section by presenting a relation with what might be called polyphase operators because of their analogy with the polyphase matrices for the filter banks of Subsec. 1.6.2. Define for \( l \in \mathbb{Z} \) the operator \( P(l) \) by
\[ P(l) : w \in L^2([0,a)) \rightarrow \left( \int_0^a g_m(t - la) w(t) \, dt \right)_{m \in \mathbb{Z}} , \]
(1.2.67)
and put
\[ P_g(e^{2\pi i l}) = \sum_{l=-\infty}^{\infty} P(l) e^{2\pi i l} , \quad \text{a.e. } \nu \in \mathbb{R} . \]
(1.2.68)
Thus for \( w \in L^2([0,a)) \)
\[ P_g(e^{2\pi i l}) w = \left( \int_0^a (Z_a g_m)(s, \nu) \, w_a(s) \, ds \right)_{m \in \mathbb{Z}} . \]
(1.2.69)
Here \( Z_a \) is a Zak transform, defined for \( h \in L^2(\mathbb{R}) \) by
\[ (Z_a h)(s, \nu) = a^{1/2} \sum_{l=-\infty}^{\infty} h(a(s - l)) e^{2\pi i l} , \quad \text{a.e. } s, \nu \in \mathbb{R} , \]
(1.2.70)
see Sec. 1.5, and \( w_a(s) = a^{1/2} w(as) \). Now \( P_g \) and \( H_g \) are related to one another according to
\[ H_g^T(\nu) = a^{1/2} P_g(e^{2\pi i a \nu}) D(\nu) F_{a^{-1}}^{-1} , \]
(1.2.71)
with \( F_{a^{-1}}^{-1} \) and \( D(\nu) \) the unitary operators defined by
\[ F_{a^{-1}}^{-1} : \zeta \in L^2(\mathbb{Z}) \rightarrow \sum_{k=-\infty}^{\infty} e^{2\pi i k l/a} x_k \in L^2([0,a)) , \]
(1.2.72)
\[ D(\nu) : w \in L^2([0,a)) \rightarrow e^{-2\pi i l \nu} w(t) \in L^2([0,a)) . \]
(1.2.73)

1.3 Weyl-Heisenberg systems as shift-invariant systems

In this section we apply the results of Sec. refs:2 to shift-invariant systems of the Weyl-Heisenberg type in which we have
\[ g_m(t) = e^{2\pi im b t} g(t) , \quad t \in \mathbb{R} , \quad m \in \mathbb{Z} , \]
(1.3.1)
with \( g \in L^2(\mathbb{R}) \) and \( b > 0 \) (the factor \( \exp(-2\pi i n m b \nu) \) is therefore included in \( g_{nm}(t) \), see (1.1.17), but does not play a significant role).

1.3.1 Frequency-domain results

Since
\[ \mathcal{F}(e^{2\pi i m b \nu} h(t)) = \hat{h}(\nu - mb) , \quad \text{a.e. } \nu \in \mathbb{R} , \quad m \in \mathbb{Z} , \]
(1.3.2)
when \( h \in L^2(\mathbb{R}) \), we see that two Weyl-Heisenberg systems \( g_{nm} , \quad (n, m) \in \mathbb{Z}^2 \), and \( \gamma_{nm} , \quad (n, m) \in \mathbb{Z}^2 \), both having a finite frame upper bound, are dual if and only if
\[ \sum_m \hat{g}^*(\nu - mb - k/a) \hat{\gamma}(\nu - mb) = a \delta_{k0} , \quad \text{a.e. } \nu \in \mathbb{R} , \quad k \in \mathbb{Z} . \]
(1.3.3)

Observing that the left-hand side of (1.3.3) is periodic in \( \nu \) with period \( b \), one can show, by computing Fourier coefficients, that the two systems (when both have a finite frame upper bound) are dual if and only if
\[ \langle \gamma, U_{k0} g \rangle = ab \delta_{k0} \delta_{l0} , \quad k, l \in \mathbb{Z} , \]
(1.3.4)
where
\[(U_{kl} h)(t) = e^{2\pi i ut/k} h(t - k/l), \text{ a.e. } t \in \mathbb{R}, \ (k,l) \in \mathbb{Z}^2, \]  
(1.3.5)
for \(h \in L^2(\mathbb{R})\). Formula (1.3.4) is the well-known Wexler-Raz biorthogonality condition for dual Weyl-Heisenberg systems, see [WR90, Jan94b, Prop. A, [Jan95b], (1.12)], [DLL95], Sec. 3, [RS95c], Subsec. 3.3.

An important consequence of (1.3.4) is the following result: the Weyl-Heisenberg system \(g_{n,m}, (n,m) \in \mathbb{Z}^2\), can be a frame only if \(ab \leq 1\). Indeed, when the system is a frame, then it follows from minimality, see (1.1.14), that \(|\langle g, \gamma \rangle| \leq 1\) by comparing the two expansions
\[g = \sum_{n,m} (g, \gamma_{n,m}) g_{n,m} = \sum_{n,m} \delta_{n0} \delta_{m0} g_{n,m}.\]  
(1.3.6)

And then (1.3.4) with \(l = k = 0\) shows that \(ab \leq 1\). We note that this argument, which was presented in [Jan94b], end of Sec. 1, has been extended in [Jannr] to more general shift-invariant systems \(g_{n,m}, (n,m) \in \mathbb{Z}^2\), where the \(g_{n,m}, m \in \mathbb{Z}\), have certain frequency localization properties.

As a consequence of Theorem 2.4 we have for the Weyl-Heisenberg system \(g_{n,m}, (n,m) \in \mathbb{Z}^2\), the frame operator representation
\[S_g f(v) = \frac{1}{a} \sum_k d_k(v) f(v - k/a)\]  
(1.3.7)
when \(f \in L^2(\mathbb{R})\) with absolute convergence for a.e. \(\nu \in \mathbb{R}\). Here
\[d_k(v) = \sum_m \hat{g}(v - mb) \hat{\gamma}^*(v - mb - k/a).\]  
(1.3.8)

We observe that the convergence of the right-hand side of (1.3.7) is in \(L^2(\mathbb{R})\)-sense when there is a \(c > 0\) such that
\[\sum_k \text{ ess sup}_{\nu \in [0,v)} |\hat{g}(\nu - kc)| < \infty\]  
(1.3.9)
(Wiener amalgam space, see [Wal92], Sections 1 and 2). The representation (1.3.7) is also known as the Walnut representation of frame operators.

As a consequence of Theorem 2.5 we have that the Weyl-Heisenberg system \(g_{n,m}, (n,m) \in \mathbb{Z}^2\), is a frame with frame bounds \(A, B\) if and only if
\[a A I \leq H_g(v) H_g^*(v) \leq a B I, \text{ a.e. } \nu \in \mathbb{R},\]  
(1.3.10)
with
\[H_g(v) = (\hat{g}(v - mb - k/a))_{k \in \mathbb{Z}, m \in \mathbb{Z}}, \text{ a.e. } \nu \in \mathbb{R}.\]  
(1.3.11)

The minimal dual system \(\gamma_{n,m}, (n,m) \in \mathbb{Z}^2\), can be computed according to Theorem 2.6 as
\[\hat{\gamma}(\nu) = \frac{1}{a} \sum_k H_g(\nu) H_g^*(\nu) \hat{g}(\nu - k/a), \text{ a.e. } \nu \in \mathbb{R}.\]  
(1.3.12)
Moreover, Theorem 2.6 shows that for any other dual system \(\gamma_{n,m}, (n,m) \in \mathbb{Z}^2\), there holds for a.e. \(\nu \in \mathbb{R}\)
\[\sum_{m=-\infty}^{\infty} |\hat{\gamma}(\nu - mb)|^2 \leq \sum_{m=-\infty}^{\infty} |\gamma(\nu - mb)|^2,\]  
(1.3.13)
with equality if and only if \(\hat{\gamma}(\nu - mb) = \gamma(\nu - mb)\) for \(m \in \mathbb{Z}\). By integration of (1.3.13) over \(\nu \in [0,b]\) and using Parseval’s theorem we see that
\[||\hat{\gamma}||^2 \leq ||\gamma||^2\]  
(1.3.14)
with equality if and only if \(\hat{\gamma} = \gamma\) a.e. Hence the minimal dual \(\hat{\gamma}\) has the least energy among all duals \(\gamma\).

We conclude this subsection by considering the case \(ab = 1\) in some more detail (this case is called the critically sampled case since a Weyl-Heisenberg system can only be a frame when \(ab \leq 1\)). In that case the operator \(H_g(\nu)\) in (1.3.11) is Toeplitz, and so is \(H_g^*(\nu) H_g(\nu)\) in (1.10). There holds
\[\langle H_g(\nu) H_g^*(\nu) \rangle_{k,l} = \sum_{m=-\infty}^{\infty} \hat{g}(\nu - (m + k) b) \hat{\gamma}^*(\nu - (m + l) b), \text{ a.e. } \nu \in \mathbb{R}.\]  
(1.3.15)
It follows from the theory of Toeplitz operators of \(L^2(\mathbb{Z})\) that the spectrum of \(H_g(\nu) H_g^*(\nu)\) is contained in the interval \([m(\nu), M(\nu)]\) with
\[m(\nu) = \text{ess inf}_{\nu} F_g(\nu), M(\nu) = \text{ess sup}_{\nu} F_g(\nu),\]  
(1.3.16)
where
\[F_g(\nu) = \sum_{k=\infty}^{\infty} \frac{1}{\hat{g}(\nu - mb) e^{2\pi i mb}} = \frac{1}{\sum_{m=-\infty}^{\infty} \hat{g}(\nu - mb) e^{2\pi i mb}}, \text{ a.e. } \nu \in \mathbb{R}.\]  
(1.3.17)
Here we need \(\nu \in \mathbb{R}\) such that \(\sum_m |\hat{g}(\nu - mb)|^2 < \infty\) (this holds for a.e. \(\nu \in \mathbb{R}\)). When there are \(m > 0, M < \infty\) such that
\[m \leq \nu(\nu) \leq M(\nu) \leq M, \text{ a.e. } \nu \in \mathbb{R},\]  
(1.3.18)
we have that \(g_{n,m}, (n,m) \in \mathbb{Z}^2\), is a frame, and \(H_g(\nu)\) is invertible for a.e. \(\nu \in \mathbb{R}\), so that there is only one dual \(\gamma\).
1.3.2 Time-domain results

The results of Sec. 1.3.1 have consequences in the time-domain as well since

\[ F[\hat{e}^{2\pi inb(t-na)}g(t-na)](v) = e^{-2\pi inav}\hat{g}(v-mb) \quad (1.3.19) \]

Thus (ignoring factors exp(2\pi inmb)) when \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), is a Weyl-Heisenberg system with shift parameters \( a, b \), then \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), is a system with shift parameters \( a, b \). In particular, we have that the systems \( g_{nm}, (n, m) \in \mathbb{Z}^2 \), and \( \gamma_{nm}, (n, m) \in \mathbb{Z}^2 \), are dual (for the shift parameters \( a, b \)) if and only if

\[ \sum_n g(t+na-l/b)\gamma^*(t+na) = b\delta_t, \text{ a.e. } t \in \mathbb{R}, l \in \mathbb{Z}. \quad (1.3.20) \]

And the minimal dual \( ^\circ\gamma \) can be computed as

\[ ^\circ\gamma(t) = b \sum_l \left( M_{l}(t)M_{l}^*(t)\right)^{-1}g(t-l/a), \text{ a.e. } t \in \mathbb{R} \quad (1.3.21) \]

where

\[ M_{l}(t) = (g(t+na-l/b))_{l \in \mathbb{Z}, n \in \mathbb{Z}}. \quad (1.3.22) \]

Let us consider the problem of computing dual \( \gamma \)'s in the case that \( g \) has bounded support. We ask ourselves whether it is possible to have a dual \( \gamma \) with bounded support as well. In the case of critical sampling \( (ab = 1) \) this is only possible when \( M_{l}(t) \) (a Toeplitz operator) is a diagonal matrix for a.e. \( t \in \mathbb{R} \). Indeed we have then that \( M_{l}(t) \) and \( M_{l}^*(t) \) are each other's inverse, and banded Toeplitz operators of \( l^2(\mathbb{Z}) \) have banded inverses, except in the case of diagonal matrices. When, however, \( ab < 1 \) the situation is different.

Consider the case that \( b^{-1} = qa \) where \( q \in \mathbb{N}, q \geq 2 \), and that \( g \in L^2(\mathbb{R}) \) is supported by an interval \( [0, ra] \) with \( r \in \mathbb{N} \). We try to satisfy (1.3.20) with a \( \gamma \) supported by \( [ua, va] \) where \( u \leq 0, v \geq r \). Observe that we need to satisfy (1.3.20) for \( 0 \leq t < a \) only by periodicity of the left-hand side. Now for \( t \in [0, a) \) we have

\[ g(t+na-lqa)\gamma^*(t+na) = 0, \quad n \in \mathbb{Z}, \quad (1.3.23) \]

due to the condition (1.3.20) reduces for \( t \in [0, a) \) to a linear system of

\[ z - w + 1; \quad z = \left[ \frac{v-1}{q} \right], \quad w = -\left[ \frac{r-1-u}{q} \right] \quad (1.3.24) \]

equations in the \( v-u \) unknowns \( \gamma(t+na), n = u, ..., v-1 \). Here we have denoted \( \lfloor x \rfloor = \text{largest integer} \leq x \). When the number in (1.3.24) \( \leq v-u \), so that the coefficient matrix

\[ M_{l}(t; u, v) = (g(t+na-lqa))_{l, u-l+1 \leq n \leq v-1, n = u, ..., v-1} \quad (1.3.25) \]

has at least as many columns as rows, this system has a fair chance to be solvable. Schematically, the situation may be depicted as follows, where the row index \( l \) runs from top to bottom and the column index \( n \) runs from left to right:

\[ \begin{array}{cccccc}
\gamma^*(t+ua) \\
\vdots \\
0 \\
\vdots \\
0 \\
1 \\
\end{array} = \begin{array}{c}
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
0 \\
\end{array} \]

Assuming (1.3.25) has full row rank for all \( t \in [0, a) \), least-norm solutions \( ^\circ\gamma^{uv} \) are then found by letting

\[ (\gamma^{uv}(t+na))_{n=u,...,v-1} = bM_{l}^+(t; u, v)\epsilon^{uv}, \quad t \in [0, a), \quad (1.3.26) \]

with \( M_{l}^+(t; u, v) \) the generalized inverse of \( M_{l}(t; u, v) \) and \( b\epsilon^{uv} \) the right-hand side vector in the above scheme. Observe that for all dual \( \gamma^{uv} \) supported by \( [ua, va] \) we have

\[ \sum_{n=ua}^{va-1} |\gamma^{uv}(t+na)|^2 \leq \sum_{n=ua}^{va-1} |\gamma^{uv}(t+na)|^2, \quad t \in [0, a), \quad (1.3.27) \]

whence \( \|\gamma^{uv}\|^2 \leq \|\gamma^{uv}\|^2 \) by integration.

Consider the following example. Let \( a = 1, q = 2 \) (so that \( b = 1/2 \)), and let

\[ g(t) = \begin{cases} \frac{1}{2} \sqrt{3}(1 - \cos \frac{1}{2} \pi t), & 0 \leq t \leq 4, \\ 0, & \text{otherwise}, \end{cases} \quad (1.3.28) \]
so that \( g \) is the familiar raised cosine window. For \( M = 1, 2, \ldots \), denote
\[
\gamma^M = \gamma^{-2M + 2, 2M + 2}
\] (1.3.29)
with \( \gamma^{M} \) given by (1.3.26). We have displayed \( \gamma^M \) for \( M = 1, 2, 3, 4 \) in Figure 1.3.1(a)-(e).

It now happens that in the present case one can determine the non-support restricted minimal \( \gamma \) of (1.3.21) analytically, see [Jan96], Sec. 3. This yields
\[
\gamma(t + n) = \frac{3}{2\sqrt{2}} \sum_{k=\lfloor \frac{1}{2}n \rfloor - 1}^{\lfloor \frac{1}{2}n \rfloor - 1} \left( \frac{-1}{3 + 2\sqrt{2}} \right)^k g(t + n - 2k),
\] \( t \in [0, 1], \ n \in \mathbb{Z}, \) (1.3.30)
and this \( \gamma \) has been displayed in Fig. 1.3.1(e). Apparently \( \gamma^M \to \gamma \) as \( M \to \infty \).

### 1.4 Weyl-Heisenberg systems in the time-frequency domain

As in Sec. 1.3 we let \( a > 0, b > 0 \), and we consider Weyl-Heisenberg systems derived from a \( g \in L^2(\mathbb{R}) \). We use in this section the notation
\[
g_{nm}(t) = g_{na, nb}(t) = e^{2\pi i nt} g(t - na), \ t \in \mathbb{R},
\] (1.4.1)
where
\[
h_{x,y}(t) = e^{2\pi i yt} h(t - x), \ t \in \mathbb{R},
\] (1.4.2)
for \( h \in L^2(\mathbb{R}), \ x \in \mathbb{R}, \ y \in \mathbb{R} \). We also consider the time-frequency shift operators \( U_{kl} \) defined for \( k \in \mathbb{Z}, \ l \in \mathbb{Z} \) by
\[
U_{kl} h = h_{k, l/a}, \ h \in L^2(\mathbb{R}),
\] (1.4.3)
The proofs of the main results of this section can be found in [Jan95b, DLL95, RS95c], where it is noted that the approaches used in these references are quite different (indeed, [Jan95b, DLL95, RS95c] have been written independently and more or less simultaneously). We shall follow the approach in [Jan95b] which is based on what we call the Fundamental Identity below, also see [DLL95], Sec. 3. It is the only thing we prove in this section; its proof can be assembled from [Jan94b], proof of Prop. A and [Jan95b], Props. 2.3 and 2.4. The technique we use in this proof is due to Tolimieri and Orr, see [TO95], Sec. 2.

**FIGURE 1.3.1.** The least-norm dual \( \gamma^M \) supported by \([-2M + 2, 2M + 2]\) for the Weyl-Heisenberg system \( g_{na, nb}, (n, m) \in \mathbb{Z}^2 \), with \( g \) the raised cosine \( c(1 - \cos \frac{1}{2} \pi t) \chi_{[0, \frac{1}{2}]}(t) \), normed such that \( ||g|| = \frac{1}{2} \sqrt{2} \), for (a) \( M = 1 \), (b) \( M = 2 \), (c) \( M = 3 \), (d) \( M = 4 \), and (e) \( M = \infty \) for which formula (1.3.30) has been used. The author wishes to thank his colleague M. Maes for producing the figures.
1.4.1 Fundamental Identity

Let $f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \in L^2(\mathbb{R})$, and assume that at least one of $f^{(1)}, f^{(2)}$ and at least one of $f^{(3)}, f^{(4)}$ generates a Weyl-Heisenberg system for the parameters $a, b$ with a finite frame upper bound. Also assume that

$$\sum_{k,l} |\langle f^{(3)}, f^{(2)}_{k/b,l/a} \rangle| |\langle f^{(1)}, f^{(4)}_{k/b,l/a} \rangle| < \infty. \quad (1.4.4)$$

Then

$$\sum_{n,m} \langle f^{(1)}_{n,a,m}, f^{(2)}_{n,a,m} \rangle \langle f^{(3)}_{n,a,m}, f^{(4)}_{n,a,m} \rangle = \frac{1}{ab} \sum_{k,l} \langle f^{(3)}_{k/b,l/a}, f^{(4)}_{k/b,l/a} \rangle \langle f^{(1)}_{k/b,l/a}, f^{(2)}_{k/b,l/a} \rangle. \quad (1.4.5)$$

Proof: Consider the function

$$H(x,y) = \sum_{n,m} \langle f^{(1)}_{n-a,y-mb}, f^{(2)}_{n-a,y-mb} \rangle \langle f^{(3)}_{x-a,y-mb}, f^{(4)}_{x-a,y-mb} \rangle \quad (1.4.6)$$

which is periodic in $x$ and $y$ with periods $a$ and $b$, respectively. This $H$ is continuous which follows from the finite frame upper bound assumption and the fact that

$$\|f_{x,y} - f_{t,s}\| \to 0, \ (x,y) \to (t,s), \quad (1.4.7)$$

for $f \in L^2(\mathbb{R})$. Here it is also useful to note that

$$\langle f_{n-a,y-mb}, h_{n,a,m} \rangle = \langle f_{x-a,y-mb}, h_{n-a,y} \rangle e^{-2\pi i nx/a + 2\pi i ny} \quad (1.4.8)$$

for $f, h \in L^2(\mathbb{R})$, $x, y \in \mathbb{R}$, and $n, m \in \mathbb{Z}$. Therefore, $H$ has the Fourier series expansion

$$H(x,y) \sim \frac{1}{ab} \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} c_{uv} e^{-2\pi i u x/a - 2\pi i v y/b}, \quad (1.4.9)$$

where

$$c_{uv} = \int_0^a \int_0^b H(x,y) e^{2\pi i u x/a + 2\pi i v y/b} \, dx \, dy. \quad (1.4.10)$$

We shall show below that

$$c_{uv} = \langle f^{(1)}_{u-b,-u/a}, f^{(4)} \rangle \langle f^{(3)}_{f/b,-y/a}, f^{(2)} \rangle, \ u, v \in \mathbb{Z}. \quad (1.4.11)$$

From this and the assumption (1.4.4), we see that the continuous function $H$ coincides everywhere with its absolutely convergent Fourier series, and then taking $x = y = 0$ in (1.4.9) we get the result.

We now show (1.4.11). By inserting (1.4.6) into (1.4.10) and rearranging sums and integrals (which is allowed by absolute and dominated convergence of the series in (1.4.6)) we get

$$c_{uv} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f^{(1)}_{x,y}, f^{(2)} \rangle \langle f^{(3)}_{x,y}, f^{(4)} \rangle e^{2\pi i u x/a + 2\pi i v y/b} \, dx \, dy \quad (1.4.12)$$

with absolute convergence at the right-hand side. There holds

$$\langle f^{(1)}_{x,y}, f^{(2)} \rangle e^{2\pi i u x/a + 2\pi i v y/b} = \langle f^{(1)}_{x-y, -u/a}, f^{(2)}_{y-b,-u/a} \rangle \quad (1.4.13)$$

for $x, y \in \mathbb{R}$ and $u, v \in \mathbb{Z}$. And now using the formula

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle k^{(1)}_{x,y}, k^{(2)} \rangle \langle k^{(3)}_{x,y}, k^{(4)} \rangle \, dx \, dy = \langle k^{(1)}, k^{(4)} \rangle \langle k^{(3)}, k^{(2)} \rangle, \quad (1.4.14)$$

valid for $k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)} \in L^2(\mathbb{R})$, see [Fol89], Section 1.4, the formula (1.4.11) follows. This completes the proof.

\[ \blacksquare \]

1.4.2 Wexler-Raz biorthogonality condition for duality

Assume that the two Weyl-Heisenberg systems $g_{n,a,m}$, $(n,m) \in \mathbb{Z}^2$, and $\gamma_{n,a,m}$, $(n,m) \in \mathbb{Z}^2$, have finite frame upper bounds. Then the systems are dual if and only if

$$\langle \gamma_{l,b,l/a}, g_{k,b, l/a} \rangle = ab \delta_{k0} \delta_{l0}, \ k, l \in \mathbb{Z}. \quad (1.4.15)$$

A version of this result was first presented by Wexler and Raz [WR90] in 1990, and it has been applied since then extensively by many authors (among which Wexler and Raz themselves), see e.g. [Jan95b, FR94, QCL92, QC93, QC94]. A proof of the above result based on the Fundamental Identity in 1.4.1 can be found in [Jan94b]; also see Subsec. 1.3.1.

The condition (1.4.15) can be written as

$$U_g \gamma = \varrho ; \ \varrho = ab (\delta_{b0} \delta_{l0}) k, l \in \mathbb{Z}, \quad (1.4.16)$$

where $U_g$ is the linear mapping of $L^2(\mathbb{R})$ defined by

$$U_g f = ((f, g_{k,b,l/a})) k, l \in \mathbb{Z}, \ f \in L^2(\mathbb{R}). \quad (1.4.17)$$
Assuming the mapping $U_g$ to be bounded and right-invertible (equivalently, $U_g U_g^*$ is bounded and invertible), one can compute a minimal energy $\gamma = \gamma_{\infty}$ according to

$$\gamma_{\infty} = U_g^* (U_g U_g^*)^{-1} g,$$

(1.4.18)
or, more explicitly, as (see (1.4.3))

$$\gamma_{\infty} = ab \sum_{k,l} \left( (U_g U_g^*)^{-1} \right)_{kl;oo} U_{kl} g.$$

(1.4.19)

Here we have used that

$$U_g^* a = \sum_{k,l} a_{kl} U_{kl} g, \ a \in L^2(\mathbb{Z}^2).$$

(1.4.20)

Now one can ask whether the minimal dual $\gamma = S_g^{-1} g$ coincides with $\gamma_{\infty}$ of (1.4.18). This is indeed so as we see from (1.3.14). A different approach to this result was presented in [DLL95], Sec. 4 and in [DLL95], Sec. 5. Yet another proof of the identity $\gamma = \gamma_{\infty}$ as given in [Jan95b], Prop. 3.3 is based on the following result for frame bounds.

**Theorem 1.4.1** For any $A \geq 0$, $B < \infty$ there holds

$$A \|f\|^2 \leq \sum_{n,m} \langle f, g_{n,m} \rangle^2 \leq B \|f\|^2, \ f \in L^2(\mathbb{R}),$$

(1.4.21)

$$\iff$$

$$AI \leq \frac{1}{ab} U_g U_g^* \leq BI,$$

where $I$ is now the identity operator of $L^2(\mathbb{Z}^2)$.

We note that Theorem 4.1 holds for all $a, b > 0$, the most interesting case being $ab \leq 1$: for $ab > 1$ we must have $A = 0$. Observe also the following. Since $U_g$ and $U_g^*$ have the same operator norm, we have

$$\frac{1}{ab} U_g U_g^* \leq BI \iff \forall f \in L^2(\mathbb{R}) \left[ \sum_{k,l} \|f, g_{k,l} \|^2 \right] \leq ab B \|f\|^2.$$

(1.4.22)

Hence the system $g_{n,m}$, $(n,m) \in \mathbb{Z}^2$, has the frame upper bound $B_g$ if and only if the system $g_{k,l}$, $(k,l) \in \mathbb{Z}^2$, has the frame upper bound $ab B_g$. It is also seen from (1.4.22) that Theorem 4.1 is trivial for $ab = 1$.

The approach in (1.4.18) for computing a (minimal) dual $\gamma$ can be generalized as follows. When $X$ is a bounded positive definite linear operator of $L^2(\mathbb{R})$, the dual $\gamma = X \gamma$ with minimal value of $\langle X \gamma, \gamma \rangle$ is given by

$$\gamma = X^{-1} U_g^* (U_g X^{-1} U_g^*)^{-1} g.$$
1.4.4 Characterization of minimum dual

The two Weyl-Heisenberg systems $g_{na, mb}, (n, m) \in \mathbb{Z}^2$, and $\gamma_{na, mb}, (n, m) \in \mathbb{Z}^2$, are dual if and only if

$$\frac{1}{ab} U_g U_{\gamma}^* = I .$$  \hspace{1cm} (1.4.30)

Furthermore, we have that

$$ \frac{1}{ab} U_{\gamma}^* U_{\gamma} = \left( \frac{1}{ab} U_g U_g^* \right)^{-1} .$$  \hspace{1cm} (1.4.31)

when $g_{na, mb}, (n, m) \in \mathbb{Z}^2$, is a frame, so that in particular

$$ \frac{1}{ab} U_{\gamma} U_{\gamma}^* = \left( \frac{1}{ab} U_g U_g^* \right)^{-1} .$$  \hspace{1cm} (1.4.32)

The proof of this result is in [Jan95b], proof of Prop. 3.4. As a consequence of (1.4.32) we have that (when $g_{na, mb}, (n, m) \in \mathbb{Z}^2$) is a frame) the inverse frame operator $S_{\gamma}^{-1} = S_{\gamma}^{*}$ has the representation

$$ S_{\gamma} = \sum_{k,l} \left( \left( \frac{1}{ab} U_g U_g^* \right)^{-1} \right)_{kl} U_{kl} $$  \hspace{1cm} (1.4.33)

with the same precautions as in (1.4.24)–(1.4.28).

1.4.5 Connection between $H_{g}(\nu)$ and $U_{g}$

Let $\Phi$ denote the Hilbert space $L^2([0, b) \times \mathbb{Z})$, so that

$$ \varphi \in \Phi \Leftrightarrow \varphi(\nu) \in L^2(\mathbb{Z}), \nu \in [0, b) \ \&$$  \hspace{1cm} (1.4.34)

$$ ||\varphi||^2 := \int_0^b \sum_k |\varphi_k(\nu)|^2 d\nu < \infty .$$

Now let $g_{na, mb}, (n, m) \in \mathbb{Z}^2$, be a Weyl-Heisenberg system with finite frame upper bound, and define a linear operator $H_{g}$ of $\Phi$ by

$$ (H_{g} \varphi)(\nu) = H_{g}^T(\nu) \varphi(\nu) , \text{ a.e. } \nu \in [0, b) ,$$  \hspace{1cm} (1.4.35)

for $\varphi \in \Phi$. Then there holds

$$ U_g^* = \sqrt{b} \mathcal{F}^{-1} R H_g Q ,$$  \hspace{1cm} (1.4.36)

where $\mathcal{F}^{-1}$ is the inverse Fourier transform of $L^2(\mathbb{R})$, and $R$ and $Q$ are unitary operators, defined by

$$ Q : \alpha \in L^2(\mathbb{Z}) \rightarrow \left( \frac{1}{\sqrt{b}} \sum_{k=-\infty}^{\infty} \alpha_{kl} e^{2\pi i k \ell /a b} e^{-2\pi i k \ell /b} \right)_{\ell \in \mathbb{Z}} \in \Phi ,$$  \hspace{1cm} (1.4.37)

and

$$ R : \varphi \in \Phi \rightarrow \varphi_{-\lfloor \nu / b \rfloor} (\nu - \lfloor \nu / b \rfloor b) \in L^2(\mathbb{R}) ,$$  \hspace{1cm} (1.4.38)

respectively. The relation (1.4.36) between the operators $H_{g}(\nu)$, $0 \leq \nu < b$, and $U_{g}$ through unitary operators shows, for instance, that Theorem 2.5 for Weyl-Heisenberg systems, see (1.3.10)–(1.3.11), and Theorem 4.1 are really the same.

1.5 Rational Weyl-Heisenberg systems in the Zak transform domain

We consider in this section Weyl-Heisenberg systems $g_{na, mb}, (n, m) \in \mathbb{Z}^2$, for the special case that $ab$ is rational, $ab = p/q$ with $p, q \in \mathbb{N}$ and $\text{GCD}(p, q) = 1$.

Let $\lambda > 0$. We define for $h \in L^2(\mathbb{R})$ the Zak transform $Z_{\lambda} h$ by

$$ (Z_{\lambda} h)(t, \nu) = \lambda^{1/2} \sum_{k=-\infty}^{\infty} h(\lambda(t - k)) e^{2\pi i k \nu} ,$$  \hspace{1cm} (1.5.1)

where the right-hand side has to be interpreted in an $L^2_{\text{loc}}(\mathbb{R}^2)$-sense. Of the many properties of the Zak transform, see [Jan88], we mention

$$ \int_{-\infty}^{\infty} f(t) h^*(t) \ dt = \int_{0}^{1} \int_{0}^{1} (Z_{\lambda} f)(t, \nu)(Z_{\lambda} h^*)(t, \nu) \ dt \ d\nu ,$$  \hspace{1cm} (1.5.2)

$$ \lambda^{1/2} f(\lambda t) = \int_{0}^{1} (Z_{\lambda} f)(t, \nu) \ d\nu , \text{ a.e. } t \in \mathbb{R} ,$$  \hspace{1cm} (1.5.3)

$$ (Z_{\lambda} f)(t + 1, \nu) = e^{2\pi i \nu} (Z_{\lambda} f)(t, \nu) , \text{ a.e. } t, \nu \in \mathbb{R} ,$$  \hspace{1cm} (1.5.4)

$$ (Z_{\lambda} f)(t, \nu + 1) = (Z_{\lambda} f)(t, \nu) , \text{ a.e. } t, \nu \in \mathbb{R} ,$$  \hspace{1cm} (1.5.5)

$$ (Z_{\lambda} f)(t, \nu) = e^{2\pi i \nu t} (Z_{1/\lambda} f)(-\nu, t) , \text{ a.e. } t, \nu \in \mathbb{R} ,$$  \hspace{1cm} (1.5.6)

where $f, h \in L^2(\mathbb{R})$. Moreover, for any $Z \in L^2_{\text{loc}}(\mathbb{R}^2)$ such that

$$ Z(t + 1, \nu) = e^{2\pi i \nu} Z(t, \nu), Z(t, \nu + 1) = Z(t, \nu) ,$$  \hspace{1cm} a.e. $t, \nu \in \mathbb{R} ,$$  \hspace{1cm} (1.5.7)
there is a unique \( f \in L^2(\mathbb{R}) \) such that \( Z = Z_f \). We note here that when \( Z \) in (1.5.7) is, in addition, continuous, we have \( Z(t, \nu) = 0 \) for some \((t, \nu) \in [0, 1)^2\).

The usefulness of the Zak transform for studying frame bound questions and computation of (minimal) duals for the case that \( ab \) is rational has been demonstrated notably by Zilbush and Zeevi, see [ZZ92b, ZZ93b, ZZ97b]; also see [Dau92], p. 978 and 981, [FZ95b, Jan95c, BC]. We shall make the choice \( \lambda = a^{-1} \), and suppress the subscript \( \lambda \) in \( Z_f \) so that

\[
(Z_f)(t, \nu) = b^{-1/2} \sum_{k=-\infty}^{\infty} h\left(\frac{t-k}{b}\right)e^{2\pi i \nu k}, \text{ a.e. } t, \nu \in \mathbb{R},
\]

for \( h \in L^2(\mathbb{R}) \). At the end of this section we shall indicate what the results given below would look like when the choice \( \lambda = a \) instead of \( \lambda = b^{-1} \) were made in \( Z_f \).

We set for \( f, h \in L^2(\mathbb{R}) \) and a.e. \( t, \nu \in \mathbb{R} \)

\[
\Phi^f(t, \nu) = p^{-1/2} \left( (Z_f)\left(t - \frac{p}{q}, \nu + \frac{k}{p}\right) \right)_{k=0,...,p-1; t=0,...,q-1} \quad (1.5.9)
\]

\[
A^h_{kr}(t, \nu) = \left( A^h_k(t, \nu) \right)_{k=0,...,p-1; r=0,...,q-1} = \Phi^f(t, \nu)(\Phi^h(t, \nu))^* \quad (1.5.10)
\]

Observe that by (1.5.2), (1.5.4) and (1.5.5) there holds

\[
\int_{0}^{q^{-1}} \int_{0}^{p^{-1}} \sum_{k=0}^{q-1} \sum_{l=0}^{p-1} \Phi^f_{kl}(t, \nu)(\Phi^h_{kl}(t, \nu))^* \, dt \, dv = \frac{1}{p} \langle f, h \rangle. \quad (1.5.11)
\]

Here we also note that (since \( \gcd(p, q) = 1 \)) by (1.5.4)

\[
(Z_f)\left(t - \frac{p}{q}, \nu + \frac{k}{p}\right) = e^{2\pi i \varphi} (Z_f)\left(t - \frac{p}{q}, \nu + \frac{k}{p}\right), \quad (1.5.12)
\]

where the permutation \( \pi \) of \( \{0, ..., q-1\} \) and \( \varphi = \varphi(k, l, \nu) \in \mathbb{R} \) are both independent of \( f \). For later reference we write (1.5.11) alternatively as

\[
\frac{1}{p} \langle f, h \rangle = \sum_{l=0}^{q-1} \int_{0}^{p^{-1}} \int_{0}^{q^{-1}} \langle \Phi^f_{l}(t, \nu), \Phi^h_{l}(t, \nu) \rangle \, dt \, dv, \quad (1.5.13)
\]

where \( \Phi^f_{l}(t, \nu), \Phi^h_{l}(t, \nu) \in \mathbb{C}^p \) are the \( l \)-th columns of \( \Phi^f(t, \nu) \), \( \Phi^h(t, \nu) \), and the inner product in the right-hand side integral of (1.5.13) is the inner product of \( \mathbb{C}^p \).

### 1.5.1 Frame operator representation

Assume that the system \( g_{n,a,m,b}, (n, m) \in \mathbb{Z}^2 \), has a finite frame upper bound, and let \( f \in L^2(\mathbb{R}) \). Then there holds

\[
\Phi^S_{n,a,m,b}(t, \nu) = A^{g_{n,a,m,b}}(t, \nu) \Phi^f(t, \nu), \ \text{ a.e. } t, \nu \in \mathbb{R}, \quad (1.5.14)
\]

and

\[
(S^*_f, f) = \int_{0}^{q^{-1}} \int_{0}^{p^{-1}} \langle A^{g_{n,a,m,b}}(t, \nu) \Phi^f_{l}(t, \nu), \Phi^f_{l}(t, \nu) \rangle \, dt \, dv. \quad (1.5.15)
\]

These formulas follow from (10) and (13) in [ZZ93b] after some further manipulations.

### 1.5.2 Frame bounds

For any \( A \geq 0, B < \infty \) there holds

\[
A \|f\|^2 \leq \sum_{n,m} \| (f, g_{n,a,m,b}) \|^2 \leq B \|f\|^2, \quad f \in L^2(\mathbb{R}), \quad (1.5.16)
\]

\[\Leftrightarrow \quad A I_{p \times p} \leq A^{g_{n,a,m,b}}(t, \nu) \leq B I_{p \times p}, \quad t, \nu \in \mathbb{R}. \]

This is shown in [ZZ93b], and it also readily follows from (1.5.13) and (1.5.15). Note that for checking the right-hand side member of (1.5.16) one can restrict oneselves to \( t \in [0, q^{-1}], \nu \in [0, p^{-1}] \). Explicitly,

\[
A^{g_{n,a,m,b}}(t + q^{-1}, \nu) = F A^{g_{n,a,m,b}}(t, \nu) F^{-1}, \quad (1.5.17)
\]

\[
A^{g_{n,a,m,b}}(t, \nu + p^{-1}) = J A^{g_{n,a,m,b}}(t, \nu) J^{-1}, \quad (1.5.18)
\]

where \( F = \text{diag}(\exp(-2\pi i n_0/k), k = 0, ..., p-1) \),

### 1.5.3 Fourier series expansion of matrix elements

Let \( f, h \in L^2(\mathbb{R}) \). Then \( A^h_{kr}(t, \nu) \) is \( p/q \)-periodic in \( t \) and \( 1 \)-periodic in \( \nu \) with Fourier series

\[
A^h_{kr}(t, \nu) \sim \sum_{n,m} d_{nmkr} e^{-2\pi i nt/p} e^{-2\pi im\nu}, \quad (1.5.19)
\]
where
\[
d_{nmkr} = \begin{cases} \frac{q}{p} e^{-2\pi i mn/p} (f, h_{m-b,n-a}) & \text{if } \nu \equiv (r-k) \mod p, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5.20}
\]

This result has been proved in [Jan95c], see Prop. 1.1.

### 1.5.4 Wexler-Raz biorthogonality condition in the Zak transform domain

Assume that \( g, \gamma \in L^2(\mathbb{R}) \). Then the two systems \( g_k, \gamma_l, (k, l) \in \mathbb{Z}^2 \), and \( \gamma_k, \gamma_l, (k, l) \in \mathbb{Z}^2 \), are biorthogonal, see (1.4.15), if and only if
\[
A^\gamma(t, \nu) = \Phi^\gamma(t, \nu) (\Phi^\gamma(t, \nu))^* = I_{p \times p}, \text{ a.e. } t, \nu \in \mathbb{R}. \tag{1.5.21}
\]
This is an immediate consequence of 5.3.

### 1.5.5 Characterization of minimal dual

Assume that the system \( g_{na, mb}, (n, m) \in \mathbb{Z}^2 \), is a frame. Then there holds
\[
(\Phi^\gamma(t, \nu))^* = (\Phi^\gamma(t, \nu))^* A^{\gamma}(t, \nu)^{-1}, \text{ a.e. } t, \nu \in \mathbb{R}, \tag{1.5.22}
\]
and in particular
\[
A^{\gamma*\gamma}(t, \nu) = (A^{\gamma}(t, \nu))^{-1}, \text{ a.e. } t, \nu \in \mathbb{R}. \tag{1.5.23}
\]
This result readily follows from 5.1 and the fact that \( S_g^{\gamma*\gamma} = g \).

### 1.5.6 Condition A for rational Weyl-Heisenberg systems

We recall that a \( g \in L^2(\mathbb{R}) \) satisfies condition A when
\[
\sum_{k,l} |(g, g_{k,l})| < \infty. \tag{1.5.24}
\]

When condition A holds, the system \( g_{na, mb}, (n, m) \in \mathbb{Z}^2 \), has a finite frame upper bound, and the frame operator representation (1.4.24) is unconditional. There holds
\[
g \in L^2(\mathbb{R}) \text{ satisfies condition A} \quad \iff \quad A^{\gamma}(t, \nu) \text{ has an absolutely convergent Fourier series.} \tag{1.5.25}
\]

Furthermore, when the system \( g_{na, mb}, (n, m) \in \mathbb{Z}^2 \), is a frame, where \( g \in L^2(\mathbb{R}) \) satisfies condition A, then \( \gamma \) satisfies condition A as well. This has been proved in [Jan95c]. It is not known whether the latter result continues to hold when \( ab \) is irrational.

#### 1.5.7 Different choice of the Zak transform

We briefly indicate what the results of this section look like when we choose \( \lambda = a, \) rather than \( b^{-1}, \) in (1.5.1). Denote for \( f, h \in L^2(\mathbb{R}) \)
\[
\Psi^f(t, \nu) = p^{-1/2} \left( \left( Z_a f \right) \left( t - \frac{k}{p}, \nu - \frac{l}{q} \right) \right)_{k=0,\ldots,p-1; l=0,\ldots,q-1}, \tag{1.5.26}
\]
\[
B^f_{kr}(t, \nu) = \left( B^f_{kr}(t, \nu) \right)_{k,r=0,\ldots,p-1} = \Psi^f(t, \nu) (\Psi^h(t, \nu))^*, \tag{1.5.27}
\]
Then there holds
\[
\Psi^{\gamma*\gamma}(t, \nu) = B^{\gamma*\gamma}(t, \nu) \Psi^f(t, \nu), \text{ a.e. } t, \nu \in \mathbb{R}. \tag{1.5.28}
\]
Also, the matrix elements \( B^f_{kr}(t, \nu) \) are 1-periodic in \( t \) and \( q^{-1}-\)periodic in \( \nu, \) with Fourier series
\[
B^f_{kr}(t, \nu) \sim \sum_{n,m} e^{2\pi i nt - 2\pi im\nu}, \tag{1.5.29}
\]
where
\[
e^{nmkr} = \frac{q}{p} e^{2\pi i mn/ab} (f, h_{\frac{-n}{1-\frac{m}{a}}, \frac{1}{2}}). \tag{1.5.30}
\]
From these two results the main results of this section can be rederived.

### 1.6 Time-discrete Weyl-Heisenberg systems

In this section we consider shift-invariant systems and Weyl-Heisenberg systems for the time-discrete case. That is, we consider systems
\[
g_{nm} = (g_{m-j-nN})_{j \in \mathbb{Z}}, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}, \quad i = \{0, \ldots, M-1\}, \tag{1.6.1}
\]
with \( N \in \mathbb{N}, \) \( M \in \mathbb{N} \) and \( g_{m} \in L^2(\mathbb{Z}), \) \( m \in \mathbb{I}, \) where for the special case of Weyl-Heisenberg systems we take
\[
g_{m} = (e^{2\pi i m/M} g(j))_{j \in \mathbb{Z}}, \quad m \in \mathbb{I}, \tag{1.6.2}
\]
with \( g \in l^2(\mathbb{Z}) \). We shall again be somewhat careless about the phase factors in \( g_{nm} \) for Weyl-Heisenberg systems. As in Secs. 1.2–1.5 we consider for these systems the duality condition, frame operator representations, frame bounds, and the characterization and computation of minimal duals in various domains. Furthermore, we present a link with the theory of filter banks.

The developments for the time-discrete case and time-continuous case are to a large extent the same. Nevertheless, we found it necessary to list the time-discrete versions of the results in detail since the correct formulation of them, taking due account of the discretization, is often a non-trivial matter. We omit, however, all proofs since most of them can be found in the existing literature, to which we refer when possible, or consist of a straightforward repeating of the arguments used for the time-continuous case once the correct time-discrete version has been formulated.

### 1.6.1 Time-discrete shift-invariant systems

We consider \( l^2(\mathbb{Z}) \) with the inner product norm \( \|f\| = \langle f, f \rangle^{1/2} \) where

\[
\langle f, h \rangle = \sum_{j=-\infty}^{\infty} f(j) h^*(j) , \quad f, h \in l^2(\mathbb{Z}).
\]  

(1.6.3)

Also, for \( l \in \mathbb{Z} \), we let \( T_l \) be the time-shift operator defined on \( l^2(\mathbb{Z}) \) by

\[
T_l f = (f(j+l))_{j \in \mathbb{Z}} , \quad f \in l^2(\mathbb{Z}).
\]  

(1.6.4)

Finally, we denote for \( h \in l^2(\mathbb{Z}) \) by \( \hat{h} \) the Fourier transform

\[
\hat{h}(\vartheta) = \sum_{j=-\infty}^{\infty} h(j) e^{-2\pi i j \vartheta} , \quad \text{a.e. } \vartheta \in \mathbb{R}.
\]  

(1.6.5)

**Proposition 1.6.1** Assume that \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), and \( \gamma_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), are two shift-invariant systems with finite frame upper bounds \( B_g, B_\gamma \) (no duality assumption). Then

\[
\sum_{m} |\hat{g}_m(\vartheta)|^2 \leq N B_g , \quad \sum_{m} |\hat{\gamma}_m(\vartheta)|^2 \leq N B_\gamma , \quad \text{a.e. } \vartheta \in \mathbb{R}.
\]  

(1.6.6)

When furthermore \( f, h \in l^2(\mathbb{Z}) \), then the function

\[
\rho(f, h)(l) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle T_l f, g_{nm} \rangle \langle \gamma_{nm}, T_l h \rangle
\]  

(1.6.7)

is periodic in \( l \in \mathbb{Z} \) with period \( N \), and has a Fourier series

\[
\rho(f, h)(l) = \sum_{k=0}^{N-1} c_k e^{-2\pi i kl/N} ,
\]  

(1.6.8)

\[
c_k = \frac{1}{N} \int_{0}^{1} \hat{f}(\vartheta) \hat{h}^*(\vartheta + k/N) \sum_{m} \hat{g}_m^*(\vartheta) \hat{\gamma}_m(\vartheta + k/N) d\vartheta , \quad k \in \mathbb{Z}.
\]  

(1.6.9)

**Theorem 1.6.2** Assume that the systems \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), and \( \gamma_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), have finite frame upper bounds. Then the systems are dual in the sense that

\[
(f, h) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle f, g_{nm} \rangle \langle \gamma_{nm}, h \rangle , \quad f, h \in l^2(\mathbb{Z}),
\]  

(1.6.10)

if and only if

\[
\sum_{m=0}^{M-1} \hat{g}_m(\vartheta - k/N) \hat{\gamma}_m(\vartheta) = N \delta_{k0} , \quad \text{a.e. } \vartheta \in \mathbb{R}, \quad k \in \mathbb{Z}.
\]  

(1.6.11)

**Proposition 1.6.3** Assume that the system \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), has a finite frame upper bound \( B_g \). Then the matrix

\[
H_g(\vartheta) := (\hat{g}_m(\vartheta - k/N))_{k=0,\ldots,N-1;m=0,\ldots,M-1}
\]  

defines for a.e. \( \vartheta \in \mathbb{R} \) a bounded linear mapping of \( C^M \) into \( C^M \) with norm \( \leq (NB_g)^{1/2} \). Explicitly, we have for a.e. \( \vartheta \in \mathbb{R} \)

\[
\sum_{k=0}^{N-1} \sum_{m=0}^{M-1} \hat{g}_m(\vartheta - k/N) \hat{\gamma}_m(\vartheta) \leq N B_g \|\beta\|^2 , \quad \beta \in C^M,
\]  

(1.6.12)

where \( \|\beta\| = (\sum_{m=0}^{M-1} |\beta_m|^2)^{1/2} \) is the norm of \( \beta \in C^M \).

**Theorem 1.6.4** Assume that the system \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), has a finite frame upper bound, and let \( f \in L^2(\mathbb{R}) \). Then we have

\[
\delta \frac{d}{d\vartheta} f(\vartheta) = \frac{1}{N} \sum_{k=0}^{N-1} d_k(\vartheta) f(\vartheta - k/N) , \quad \text{a.e. } \vartheta \in \mathbb{R},
\]  

(1.6.14)
where, see (1.6.12),
\[
d_k(\vartheta) = \left( H_\vartheta(\vartheta) H_\vartheta^*(\vartheta) \right)_{\vartheta k} = \sum_{m=0}^{M-1} \hat{g}_m(\vartheta) \hat{g}_m^*(\vartheta - k/N) , \text{ a.e. } \vartheta \in \mathbb{R} ,
\]
for \( k = 0, ..., N - 1 \).

**Theorem 1.6.5** Let \( g_m \in l^2(\mathbb{R}) \), \( m \in \mathcal{I} \), and let \( A \geq 0, B < \infty \). Then
\[
A \|f\|^2 \leq \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} |(f, g_{nm})|^2 \leq B \|f\|^2 , \quad f \in l^2(\mathbb{Z})
\]
\[
\Leftrightarrow \quad N A I \leq H_\vartheta(\vartheta) H_\vartheta^*(\vartheta) \leq N B I , \quad \text{ a.e. } \vartheta \in \mathbb{R} ,
\]
where \( I \) denotes the identity mapping of \( \mathbb{C}^N \).

As a consequence of this result we see that the system \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), can only be a frame when \( N \leq M \) since \( H_\vartheta(\vartheta) \) has rank \( \leq M \).

**Theorem 1.6.6** Assume that the system \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), is a frame, and denote by \( c(\vartheta) \in \mathbb{C}^M \) for a.e. \( \vartheta \in \mathbb{R} \) the least-norm solution \( c = (c_m)_{m=0, ..., M-1} \) of the linear system
\[
\sum_{n=0}^{M-1} \hat{g}_m(\vartheta - k/N) c_m(\vartheta) = N \delta_{k0} , \quad k = 0, ..., N - 1 .
\]

Then there holds
\[
\tilde{c}_m(\vartheta) = \hat{g}_m(\vartheta) , \quad m \in \mathcal{I} , \quad \text{ a.e. } \vartheta \in \mathbb{R} .
\]

**Theorem 1.6.7** Assume that the systems \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), and \( \gamma_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), have finite frame upper bounds. Then they are dual if and only if
\[
H_\vartheta(\vartheta) H_\vartheta^*(\vartheta) = N I , \quad \text{ a.e. } \vartheta \in \mathbb{R} .
\]

Moreover,
\[
H_\vartheta^*(\vartheta) = N H_\vartheta^*(\vartheta) \left( H_\vartheta(\vartheta) H_\vartheta^*(\vartheta) \right)^{-1} , \quad \text{ a.e. } \vartheta \in \mathbb{R} ,
\]
so that in particular
\[
\frac{1}{N} H_{\vartheta}(\vartheta) H_{\vartheta}^*(\vartheta) = \left( \frac{1}{N} H_\vartheta(\vartheta) H_\vartheta^*(\vartheta) \right)^{-1} , \quad \text{ a.e. } \vartheta \in \mathbb{R} .
\]

As a consequence of (1.6.21) we note that, see Theorem 6.2, the inverse frame operator \( S_{\gamma}^{-1} = S_{\vartheta} \gamma \) has the representation
\[
\tilde{S}_{\gamma}(\vartheta) f(\vartheta) = \sum_{k=0}^{N-1} \left( \frac{1}{N} H_\vartheta(\vartheta) H_\vartheta^*(\vartheta) \right)^{-1} \hat{f}(\vartheta - k/N) , \quad \text{ a.e. } \vartheta \in \mathbb{R} ,
\]
where \( f \in l^2(\mathbb{Z}) \).

The proofs of the results in this section consist of simple adaptations of the proofs in Sec. 1.2 for the corresponding time-continuous versions of the results, except that certain measure theoretic intricacies are absent now due to finite summation ranges.

### 1.6.2 Time-discrete shift-invariant systems and filter banks

Time-discrete shift-invariant dual systems can be considered as, what are called in engineering literature, perfect reconstructing (PR) filter banks of the FIR- or IIR-type according as the impulse responses \( g_m, \gamma_m \) are finite or infinite. In this view the formula expressing duality,
\[
f = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle f, \gamma_{nm} \rangle g_{nm} = T_{\gamma} T_{\vartheta} f , \quad f \in l^2(\mathbb{Z}) ,
\]
see (1.1.3), (1.1.5)–(1.1.7), can be regarded as a representation of time-discrete signals \( f \) by means of the expansion coefficients (subband signals) \( \langle f, \gamma_{nm} \rangle, (n,m) \in \mathbb{Z} \times \mathcal{I} \). Thus the finite-energy input signal \( f \) is passed through analysis filters (A) with impulse responses \( \gamma_m(-j) \) \( j \in \mathbb{Z}, m \in \mathcal{I} \), whose outputs
\[
\sum_{j=-\infty}^{\infty} f(j) \gamma_m^*(j - k) , \quad k \in \mathbb{Z} ,
\]
are downsampled (DS) by a factor \( N \), so that only the samples with \( k = nN, n \in \mathbb{Z} \), are retained. This operation is usually followed by a quantization step (Q), in which the outputs \( \langle f, \gamma_{nm} \rangle \) are quantized to numbers \( [f, \gamma_{nm}] \), and a coding operation (C), after which the resulting data are transmitted or stored. To regenerate a distorted version \( [f] \) of the input signal, one has to decode the data, and to apply synthesis filters (S) with impulse responses \( g_m, m \in \mathcal{I} \), to upsampled versions of the numbers \( [f, \gamma_{nm}] \). The upsampling operation (US) just amounts to inserting \( N - 1 \) zeros between any two consecutive numbers \([f, \gamma_{nm}]\) with \( n \in \mathbb{Z} \). Hence
\[
[f](j) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} [f, \gamma_{nm}] g_{m}(j - nN) .
\]
Schematically one has the situation as depicted in Fig. 1.6.1.

When the QCD-operation in Fig. 1.6.1 does not introduce distortion, so that

\[ [(f, \gamma_{nm})] = (f, \gamma_{nm}) \quad n, m \in \mathbb{Z} \times \mathcal{I} \]

the formulas (1.6.23), (1.6.25) show that the systems \( \gamma_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), should be dual so as to achieve perfect reconstruction for all \( f \in L^2(\mathbb{Z}) \). (Note that for the notion of duality we may interchange the roles of the \( \gamma \)'s and the \( g \)'s.) When the QCD-operation is imperfect, the question naturally arises how to find a given \( \beta \in L^2(\mathbb{Z} \times \mathcal{I}) \) an \( f \in L^2(\mathbb{Z}) \) with minimal value of

\[ \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} |(f, \gamma_{nm}) - \beta_{nm}|^2. \]

It turns out that the minimizing \( f \) is given by

\[ f = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \beta_{nm} \circ g_{nm} \]

with \( \circ g_{nm} = S_\gamma^{-1} \gamma_m \), the minimal duals corresponding to \( \gamma_m, m \in \mathcal{I} \) (again the roles of the \( g \)'s and the \( \gamma \)'s have been interchanged). Hence the relevance of minimal dual systems for filter banks.

In data-transmission and/or storage systems one aims at signal representation methods that allow easy implementation while being efficient and robust and that do not introduce perceptually annoying distortion in the processed signals. For the present case one is thus faced with several constraints, in addition to the perfect reconstruction property, on \( N, M \), and the analysis and synthesis filters. Below we list a number of these constraints where it is noted that a particular constraint can be severe for one application and unimportant for another.

1.6.2.1 Implementational constraints

For easy implementation and a fast signal processing it is desirable that \( M \) is not too large, that the \( g_m \) and \( \gamma_m, m \in \mathcal{I} \), are of short duration, and that \( N \) is large. Also, it is convenient that the \( g_m \) and \( \gamma_m, m \in \mathcal{I} \), are real and all of the same form, or perhaps also that \( g_m = \gamma_m, m \in \mathcal{I} \). Furthermore, an integer relation between \( N \), \( M \) and the durations of the \( g_m \) and \( \gamma_m, m \in \mathcal{I} \), is desirable.

1.6.2.2 Coding-efficiency constraints

Coding efficiency is increased when advantage is taken of the masking effects occurring in the processing of stimuli by human observers. For certain applications this means that the analysis filter should have a reasonable amount of frequency selectivity, so that quantization of the outputs \( (f, \gamma_{nm}) \) can take place in accordance with their relative amplitudes. It may also be desirable that all but one pair \( \gamma_m, g_m \) satisfy \( \hat{g}_m(0) = \hat{\gamma}_m(0) = 0 \), so that the DC-component in \( f \) is represented by only one term in its expansion. The coding system's complexity is low for low values of \( M \) and large values of \( N \). Using larger values of \( M \) and/or lower values of \( N \) does not necessarily decrease the coding efficiency, but then the coding system must be able to remove the redundancy thus introduced and this increases the coding system's complexity.

1.6.2.3 Perceptual constraints

To avoid blocking effects or pre-echos, the impulse responses \( g_m \) of the filter banks should be smooth, in particular at the boundaries of their supports. Consequently, because of the perfect reconstruction property, the time-shifted \( g_m \) should overlap. On the other hand, the impulse responses \( g_m \) should have short length so as to avoid (quantization) errors in the \( (f, \gamma_{nm}) \) to have an impact on large regions (ringing). To avoid phase errors, the analysis and/or synthesis filters should be linear phase (i.e. the impulse responses, assumed to be real, should be symmetric around the midpoint of their support).

1.6.2.4 Robustness constraints

To avoid (quantization) errors in the \( (f, \gamma_{nm}) \) to have large detrimental effects, or to avoid that some signals \( f \) of unit energy need large coefficients
in their expansion, it is desirable that the ratio $A/B$ of the frame bounds is not too small. Even more desirable would be $A/B = 1$ (tight frame), so that $\gamma_m = \frac{1}{A} g_m$, $m \in I$, or that $g_{nm}$, $(n,m) \in \mathbb{Z} \times I$, is an orthonormal base for $l^2(\mathbb{Z})$.

It is the designer's task to construct perfect reconstructing filter banks such that some or many of the above constraints are satisfied. For instance in [Heu96] there are designed, for the purpose of data compression of images, critically sampled $(N = M)$, perfect reconstructing, orthonormal $(g_m = \gamma_m, m \in I)$, 50% overlapped (support length of the responses equals $2N$), linear phase filters with smooth responses $g_m$ satisfying $g_m(0) = 0$ for all $m \in I$ but one, and with acceptable frequency selectivity. As one can see, the constraints on analysis and synthesis filters are formulated in different domains (time domain, frequency domain, and perhaps also in the time-frequency domain). Some of the constraints are of the hard type (such as the perfect reconstruction property and the linear phase condition) in the sense that they can be formulated as a set of equations involving the filter impulse responses in one of the domains. Other constraints are of the soft type (such as reasonable amount of frequency selectivity, smoothness). Since we have been able to formulate the most significant and most difficult constraint of the hard type, viz. perfect reconstruction, in various domains, a possible strategy would be to introduce a cost functional incorporating (some of) the soft constraints, and to minimize this functional under the condition that perfect reconstruction is satisfied.

We next present a link with, what are called in the theory of filter banks, polyphase matrices. Here one considers matrices

$$P_l = (g_m(j-lN))_{m=0,\ldots,M-1; j=0,\ldots,N-1} \quad (1.6.29)$$

and puts

$$P_g(e^{2\pi i \theta}) = \sum_{l=-\infty}^{\infty} P_l e^{2\pi i l \theta} \quad (1.6.30)$$

Thus we have

$$P_g(e^{2\pi i \theta}) = ((Z_N g_m)(j,\theta))_{m=0,\ldots,M-1; j=0,\ldots,N-1} \quad (1.6.31)$$

where $Z_N$ is a discrete Zak transform, defined for $h \in l^2(\mathbb{Z})$ by

$$(Z_N h)(j,\theta) = \sum_{l=-\infty}^{\infty} h(j-lN) e^{2\pi i l \theta}, j \in \mathbb{Z}, \text{ a.e. } \theta \in \mathbb{R}. \quad (1.6.32)$$

Also see Subsec. 1.6.5. The matrix $P_g(e^{2\pi i \theta})$ is the polyphase matrix corresponding to the $M$-channel filter bank with impulse responses $g_m$, $m \in I$, and downsampling factor $N$. Here we note that in filter bank theory it is customary to write $z = e^{2\pi i \theta}$, so that the Zak transforms in (1.6.31) become $Z$-transforms.

There holds

$$H_g(\theta) = \sqrt{N} P_g(e^{2\pi i N \theta}) D(\theta) F_N^{-1}, \text{ a.e. } \theta \in \mathbb{R}, \quad (1.6.33)$$

with $H_g(\theta)$ given in (1.6.12), and $D(\theta)$ and $F_N^{-1}$ the unitary operators

$$D(\theta) : \mathbb{C}^N \to (e^{-2\pi i \theta} x_r)_{r=0,\ldots,N-1} \in \mathbb{C}^N, \quad (1.6.34)$$

$$F_N^{-1} : \mathbb{C}^N \to \left( \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i kr/N} x_r \right)_{k=0,\ldots,N-1} \in \mathbb{C}^N. \quad (1.6.35)$$

There is an extensive theory on filter design for analysis-synthesis filter banks by using polyphase matrices, see e.g. [Mal92, Vai93, VK95]. More recently the connection between the theories of DFT filter banks and of (oversampled) Weyl-Heisenberg frames has been established, see e.g. [Cve95a, BHF96c] and Chapter 9 of this volume, while the connections between the theories of more general filter banks and of time-discrete shift-invariant systems are in the process of being established, see [CV, Jan95a], BH95).

We conclude this subsection by a short dictionary2 of terms used in polyphase filter bank theory and shift-invariant system theory.

1.6.3 Time-discrete Weyl-Heisenberg systems as shift-invariant systems

We consider now time-discrete Weyl-Heisenberg systems $g_{nm}$, $(n,m) \in \mathbb{Z} \times I$, with

$$g_m(j) = e^{2\pi i mj/M} g(j), j \in \mathbb{Z}, \quad (1.6.36)$$

and $g \in l^2(\mathbb{Z})$.

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1 The author wishes to thank R. Heusdens for introducing him to the subject of constrained filter design, and for providing him the considerations and insights that led to the above list of constraints.

2 The author wishes to thank H. Bölcskei for advice in preparing this dictionary.


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<td>the systems (g_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}), and (\gamma_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}), are dual</td>
</tr>
<tr>
<td>paraunitary filter bank</td>
<td>the system (g_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}), is a tight frame (A_g = B_g), so that (\mathcal{O}_m = c, g_m, m \in \mathcal{I})</td>
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<tr>
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<td>the system (g_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}), is a frame with (N = M) and unique dual system (\gamma_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I})</td>
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<tr>
<td>polyphase representation</td>
<td>the matrix (H_g^T(\vartheta)), see (1.6.12)</td>
</tr>
<tr>
<td>alias component matrix</td>
<td>the matrix (P_g(e^{2\pi i \vartheta})), see (1.6.31)</td>
</tr>
<tr>
<td>polyphase matrix</td>
<td>the systems (g_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}), and (\gamma_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}), are Weyl-Heisenberg systems</td>
</tr>
<tr>
<td>DFT filter bank</td>
<td>the Weyl-Heisenberg system (g_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}), is a frame with (N = M = 2), so that the duality condition takes a special form</td>
</tr>
</tbody>
</table>

### 1.6.3.1 Frequency-domain results

Since now

\[
\hat{g}_m(\vartheta) = \hat{g}(\vartheta - m/M), \quad \text{a.e. } \vartheta \in \mathbb{R}, \ m \in \mathcal{I}, \quad (1.6.37)
\]

the results of Subsec. 1.6.1 can be translated straightforwardly to results for time-discrete Weyl-Heisenberg systems. For instance, the two systems \(g_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}\), and \(\gamma_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}\), with finite frame upper bounds are dual if and only if

\[
\sum_{k=0}^{M-1} \hat{g}^*(\vartheta - m/M - k/N) \hat{\gamma}(\vartheta - m/M) = N \delta_{k_0}, \quad \text{a.e. } \vartheta \in \mathbb{R}, \ k = 0, ..., N - 1. \quad (1.6.38)
\]

Also, there is the frame operator representation result

\[
\delta_j f(\vartheta) = \frac{1}{N} \sum_{k=-\infty}^{\infty} d_k(\vartheta) f(\vartheta - k/N), \quad \text{a.e. } \vartheta \in \mathbb{R}, \quad (1.6.39)
\]

for \(f \in L^2(\mathbb{Z})\), where

\[
d_k(\vartheta) = \sum_{m=0}^{M-1} \hat{g}^*(\vartheta - m/M) \hat{\gamma}(\vartheta - m/M - k/N), \quad \text{a.e. } \vartheta \in \mathbb{R}, \ k = 0, ..., N - 1. \quad (1.6.40)
\]

The matrix \(H_g(\vartheta)\) in (1.6.12) assumes the form

\[
H_g(\vartheta) = \left(\hat{g}(\vartheta - m/M - k/N)\right)_{k=0, ..., N-1; m=0, ..., M-1}, \quad \text{a.e. } \vartheta \in \mathbb{R}, \quad (1.6.41)
\]

and the system \(g_{nm}, (n, m) \in \mathbb{Z} \times \mathcal{I}\), is a frame with frame bounds \(A_g > 0, B_g < \infty\), if and only if

\[
N A_g I \leq H_g(\vartheta) H^*_g(\vartheta) \leq N B_g I, \quad \text{a.e. } \vartheta \in \mathbb{R}, \quad (1.6.42)
\]

with \(I\) the \(N \times N\)-identity matrix. Furthermore, the minimal dual \(^\diamond \gamma\) can be computed as

\[
^\diamond \gamma(\vartheta) = \sum_{k=0}^{N-1} \left(\frac{1}{N} H_g(\vartheta) H^*_g(\vartheta)\right)^{-1}_{ek} \hat{g}(\vartheta - k/N), \quad \text{a.e. } \vartheta \in \mathbb{R}, \quad (1.6.43)
\]

and we have \(\|\gamma\| \leq \|\gamma\|\) when \(\gamma\) is another dual (equality if and only if \(\gamma = ^\diamond \gamma\) a.e.). Finally, the polyphase matrix \(P_g(e^{2\pi i \vartheta})\) of (1.6.30) assumes for a.e. \(\vartheta \in \mathbb{R}\) the form

\[
P_g(e^{2\pi i \vartheta}) = \left(e^{2\pi i \vartheta j/N}(Z_N g)(j, \vartheta - m/M)\right)_{m=0, ..., M-1; j=0, ..., N-1}, \quad (1.6.44)
\]

with \(Z_N\) the Zak transform of (1.6.32).

### 1.6.3.2 Time-domain results

The results of the previous subsection cannot be translated immediately into time-domain results. This is so since, by contrast with the time-continuous case, see Subsection 1.3.2, the (discrete) Fourier transform does not map an infinite-time, discrete Weyl-Heisenberg system onto another such system. Nevertheless, a great deal of the results of the previous subsection have time-domain counterparts.

For instance when we denote by \(j \in \mathbb{Z}\) by \(M_g(j)\) the matrix

\[
M_g(j) = (g(j + nN - IM))_{i \in \mathbb{Z}, n \in \mathbb{Z}}, \quad (1.6.45)
\]
then the system \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), is a discrete Weyl-Heisenberg frame with frame bounds \( A_g > 0, B_g < \infty \) if and only if

\[
\frac{1}{N} A_g I \leq M_g(j) M^*_g(j) \leq \frac{1}{N} B_g I, \ j = 0, \ldots, N - 1,
\]  
(1.646)

where \( I \) is the identity operator of \( l^2(\mathbb{Z}) \). And two Weyl-Heisenberg systems \( g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), and \( \gamma_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I} \), are dual if and only if

\[
\sum_{n=-\infty}^{\infty} g^*(j+nN-lM) \gamma(j+nN) = \frac{1}{M} \delta_{j,l}, \ l \in \mathbb{Z},
\]  
(1.647)

for \( j = 0, \ldots, N - 1 \). Furthermore, there is the frame operator representation result

\[
(S_g f)(j) = M \sum_{l=-\infty}^{\infty} (M_g(j) M^*_g(j))_{ol} f(j-lM), \ j \in \mathbb{Z},
\]  
(1.648)

valid for \( f \in l^2(\mathbb{Z}) \). Also, the minimal dual \( \gamma \) can be computed from the least-norm solutions \( g^*(j) = (\gamma(j+nN))_{n \in \mathbb{Z}} \) of (1.647) for \( j = 0, \ldots, N - 1, \) and there results

\[
\gamma(j+nN) = \sum_{l=-\infty}^{\infty} (M_g(j) M^*_g(j))_{ol}^{-1} g(j+nN-lM)
\]  
(1.649)

for \( j = 0, \ldots, N - 1 \) and \( n \in \mathbb{Z} \). In particular we have, when \( \gamma \) is another dual, that for \( j = 0, \ldots, N - 1 \)

\[
\sum_{n=-\infty}^{\infty} |\gamma(j+nN)|^2 \leq \sum_{n=-\infty}^{\infty} |\gamma(j+nN)|^2
\]  
(1.650)

with equality if and only if \( \gamma(j+nN) = \gamma(j+nN), \ n \in \mathbb{Z} \). Finally, duality can equivalently be expressed as

\[
M_g(j) M^*_g(j) = \frac{1}{M} I, \ j = 0, \ldots, N - 1,
\]  
(1.651)

and the minimal dual \( \gamma \) satisfies

\[
M^*_g(j) = M^*_g(j) (M M^*_g(j) M^*_g(j))^{-1}, \ j = 0, \ldots, N - 1,
\]  
(1.652)

so that in particular

\[
M M^*_g(j) M^*_g(j) = (M M^*_g(j) M^*_g(j))^{-1}, \ j = 0, \ldots, N - 1.
\]  
(1.653)

For the inverse frame operator \( S^-_g = S^*_\gamma \), the last formula has as a consequence that

\[
(S^*_\gamma f)(j) = \sum_{l=-\infty}^{\infty} \left( (M M^*_g(j) M^*_g(j))^{-1} \right)_{ol} f(j-lM), \ j \in \mathbb{Z},
\]  
(1.654)

holds for \( f \in l^2(\mathbb{Z}) \).

All these results can be proved by carefully carrying out the program of Sec. 1.2 for the present case, where we note that there are certain simplifications since there are no measure theoretic intricacies.

### 1.6.4 Discrete Weyl-Heisenberg systems in the time-frequency domain

We let for \( k \in \mathbb{Z}, \ \theta \in \mathbb{R} \)

\[
h_{k,\theta}(j) = h(j-k) e^{2\pi i j \theta}, \ j \in \mathbb{Z},
\]  
(1.655)

when \( h \in l^2(\mathbb{Z}) \). With \( M, N \) as in Subsec. 1.6.3 we use in this section the notation

\[
g_{nm}(j) = g_{nm,M/M}(j) = (j-nN) e^{2\pi imj/M}, \ j \in \mathbb{Z}
\]  
(1.656)

for \( n \in \mathbb{Z}, m = 0, \ldots, M - 1 \), where, as usual, we do not bother about the factor \( \exp(-2\pi imnN/M) \) omitted at the right-hand side. We also consider the time-frequency shift operators \( U_{kl} \) defined for \( k, l \in \mathbb{Z} \) by

\[
U_{kl} h = h_{k,M,l/N}, \ h \in l^2(\mathbb{Z})
\]  
(1.657)

The proofs of the main results in this subsection can be given by carefully mimicking the arguments presented in [Jan95b] for the time-continuous case; also, they are presented in all detail in [Jan94a], Ch. 3, where we note that also the case of discrete, periodic Weyl-Heisenberg systems is treated in [Jan94a].

#### 1.6.4.1 Fundamental Identity

Let \( f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \in l^2(\mathbb{Z}) \), and assume that at least one of \( f^{(1)}, f^{(2)} \) and at least one of \( f^{(3)}, f^{(4)} \) generates a discrete Weyl-Heisenberg system (for the parameters \( N \) and \( M \)) with a finite frame upper bound. Also assume that

\[
\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |(f^{(3)}, f^{(2)}_{k,M,l/N})| |(f^{(1)}_{k,M,l/N}, f^{(4)}_{k,M,l/N})| < \infty.
\]  
(1.658)
Then there holds
\[
\sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle f^{(1)}_{nN,m/M}, f^{(2)}_{nN,m/M} \rangle \langle f^{(3)}_{N,m/M}, f^{(4)}_{N,m/M} \rangle = \frac{M}{N} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} \langle f^{(3)}_{kM,l/M}, f^{(1)}_{kM,l/M} \rangle \langle f^{(2)}_{kM,l/M}, f^{(4)}_{kM,l/M} \rangle .
\] (1.659)

### 1.6.4.2 Wexler-Raz biorthogonality condition for duality

Assume that the two discrete Weyl-Heisenberg \( g_{nN,m/M}, (n,m) \in \mathbb{Z} \times I, \) and \( \gamma_{nN,m/M}, (n,m) \in \mathbb{Z} \times I, \) have a finite frame upper bound. Then the systems are dual if and only if
\[
(\gamma, g_{kM,l/M}) = \frac{N}{M} \delta_{k0} \delta_{l0}, \quad k \in \mathbb{Z}, \; l = 0, \ldots, N-1 .
\] (1.660)

The condition (1.660) can also be written as
\[
U_g \gamma = \alpha ; \; \alpha = \frac{N}{M} \delta_{00} \delta_{00} \delta_{l0} e_{k,1}, \quad k \in \mathbb{Z}, \; l = 0, \ldots, N-1 ,
\] (1.661)
where \( U_g \) is the linear mapping of \( l^2(\mathbb{Z}) \) defined by
\[
U_g f = (\langle f, g_{kM,l/M} \rangle)_{k \in \mathbb{Z}, l = 0, \ldots, N-1 ,} , \quad f \in l^2(\mathbb{Z}) .
\] (1.662)

### 1.6.4.3 Frame bounds in the time-frequency domain

For any \( A \geq 0, \; B < \infty \) there holds
\[
A \| f \|^2 \leq \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \| f_{nN,m/M} \|^2 \leq B \| f \|^2 , \quad f \in l^2(\mathbb{Z}) ,
\] (1.663)
\[
\Leftrightarrow \quad AI \leq \frac{M}{N} U_g U^{*}_g \leq BI
\] with \( I \) the identity operator of \( l^2(\mathbb{Z} \times \{0, \ldots, N-1\}) \). Also, the discrete Weyl-Heisenberg system \( g_{nN,m/M}, (n,m) \in \mathbb{Z} \times I, \) has the finite frame upper bound \( B_g \) if and only if the system \( g_{kM,l/M}, k \in \mathbb{Z}, \; l = 0, \ldots, N-1, \) has the frame upper bound \( N M^{-1} B_g \).

### 1.6.4.4 Frame operator representation in the time-frequency domain

Assume that the system \( g_{nN,m/M} , (n,m) \in \mathbb{Z} \times I, \) has a finite frame upper bound. Then the frame operator \( S_g \) has the representation
\[
S_g = \frac{M}{N} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{kl} U_{kl} ,
\] (1.664)
where
\[
c_{kl} = (U_g U^{*}_g)_{k,l,00} = \langle g, g_{kM,l/M} \rangle , \quad k \in \mathbb{Z}, \; l = 0, \ldots, N-1 ,
\] (1.665)
in the sense that for any \( f, h \in l^2(\mathbb{Z}) \) such that
\[
\sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} |(U_{kl} f, h)|^2 < \infty
\] (1.666)
there holds
\[
\langle S_g f, h \rangle = \frac{M}{N} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{kl} (U_{kl} f, h) .
\] (1.667)

In the case that \( g \) satisfies Tolimieri and Orr's condition \( \Lambda \),
\[
E := \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} \| (g, g_{kM,l/M}) \| < \infty ,
\] (1.668)
the system \( g_{nN,m/M} , (n,m) \in \mathbb{Z} \times I, \) has the frame upper bound \( E \), and the representation (1.664) is unconditional.

### 1.6.4.5 Computation and characterization of minimal duals

Assume that the system \( g_{nN,m/M} , (n,m) \in \mathbb{Z} \times I, \) is a frame. Then the minimal dual \( \diamond \gamma \) can be computed as
\[
\diamond \gamma = U^{*}_g (U_g U^{*}_g)^{-1} \alpha = \frac{N}{M} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} (U_g U^{*}_g)^{-1})_{k,l,00} U_{kl} g ,
\] (1.669)
see (1.661). Also, the two systems \( g_{nN,m/M}, (n,m) \in \mathbb{Z} \times I, \) and \( \gamma_{nN,m/M}, (n,m) \in \mathbb{Z} \times I, \) both having a finite frame upper bound, are dual if and only if
\[
\frac{M}{N} U_g U^{*}_g = I ,
\] (1.670)
and the minimal dual \( \diamond \gamma \) satisfies
\[
\frac{M}{N} U^{*}_{\diamond \gamma} = U^{*}_g (U_g U^{*}_g)^{-1} ,
\] (1.671)
so that in particular
\[
\frac{M}{N} U^{*}_{\diamond \gamma} U^{*}_{\diamond \gamma} = (\frac{M}{N} U_g U^{*}_g)^{-1} .
\] (1.672)
As a consequence the inverse frame operator \( S_g^{-1} = S_{\gamma} \) has the representation
\[
S_{\gamma} = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} \left( \frac{M}{N} U_g U_g^* \right)^{-1} \big|_{k00} U_{kl} \, ,
\] (1.6.73)
with the same precautions as in the previous subsection.

1.6.4.6 **Connection between \( H_g(\vartheta) \) and \( U_g \)**

Let for \( K \in \mathbb{N} \) the Hilbert space \( L^2([0, M^{-1}] \times \{0, \ldots, K - 1\}) \) be denoted by \( \Phi_K \), so that
\[
\varphi \in \Phi_K \leftrightarrow \varphi(\vartheta) \in C^K, \quad \vartheta \in [0, M^{-1}) \quad \& \quad \|\varphi\|^2 := \int_0^{1/M} \sum_{k=0}^{K-1} \left| \varphi_k(\vartheta) \right|^2 d\vartheta < \infty \, .
\] (1.6.74)

Assume that the system \( \vartheta n, m/M, (n, m) \in \mathbb{Z} \times \mathbb{Z} \), has a finite frame upper bound, and define a linear mapping \( H_g : \Phi_N \rightarrow \Phi_g \) by
\[
( H_g \varphi)(\vartheta) = H_g^2(\vartheta) \varphi(\vartheta) \, , \quad \text{a.e. } \vartheta \in [0, \frac{1}{M}) \, ,
\] (1.6.75)
for \( \varphi \in \Phi_N \). Then there holds
\[
U_g^* = \frac{1}{\sqrt{M}} F^{-1} R H_g Q \, ,
\] (1.6.76)
where \( F^{-1} \) is the inverse Fourier transform, defined by
\[
F^{-1} : H \in L^2([0, 1)) \to F^{-1} H = \left( \int_0^1 e^{2\pi ij\vartheta} H(\vartheta) d\vartheta \right) \in l^2(\mathbb{Z}) \, ,
\] (1.6.77)
and \( R \) and \( Q \) are unitary operators, defined by
\[
Q : \alpha \in l^2(\mathbb{Z} \times \{0, \ldots, N - 1\}) \to (Q \alpha)(\vartheta) = \left( \sqrt{M} \sum_{k=-\infty}^{\infty} \alpha_k e^{2\pi ikM/N} e^{-2\pi ikM\vartheta} \right) \in \Phi_N \, ,
\] (1.6.78)
and
\[
R : \psi \in \Phi_M \to \psi_{-\lfloor M \vartheta \rfloor} \left( \vartheta - M^{-1} \lfloor M \vartheta \rfloor \right) \in L^2([0, 1)) \, ,
\] (1.6.79)
respectively.

1.6.5 **Discrete Weyl-Heisenberg systems in the Zak transform domain**

We now consider discrete Weyl-Heisenberg systems in the Zak transform domain, and we let
\[
\text{LCM}(N, M) = Nq = Mp
\] (1.6.80)
with \( p, q \in \mathbb{N} \) and \( \text{GCD}(p, q) = 1 \). Observe that
\[
K := \frac{N}{p} = \frac{M}{q} \in \mathbb{Z} \, .
\] (1.6.81)

We define for \( L \in \mathbb{N} \) the discrete Zak transform \( Z_L \) by
\[
(Z_L f)(j, \vartheta) = \sum_{k=-\infty}^{\infty} f(j - kL) e^{2\pi ik\vartheta} \, , \quad j \in \mathbb{Z} \, , \quad \text{a.e. } \vartheta \in \mathbb{R} \, ,
\] (1.6.82)
where \( f \in l^2(\mathbb{Z}) \). This definition agrees with the one given in [Hei89], except that \( \vartheta \) is replaced by \(-\vartheta\). Then we have the following
\[
\sum_{j=0}^{\infty} f(j) h^*(j) = \sum_{j=0}^{\infty} \int_0^1 (Z_L f)(j, \vartheta)(Z_L h)^*(j, \vartheta) d\vartheta \
\] (1.6.83)
\[
f(j) = \int_0^1 (Z_L f)(j, \vartheta) d\vartheta \, , \quad j \in \mathbb{Z} \, ,
\] (1.6.84)
\[
(Z_L f)(j + L, \vartheta) = e^{2\pi i j \vartheta} (Z_L f)(j, \vartheta) \, , \quad j \in \mathbb{Z} \, , \quad \text{a.e. } \vartheta \in \mathbb{R} \, ,
\] (1.6.85)
\[
(Z_L f)(j, \vartheta + 1) = (Z_L f)(j, \vartheta) \, , \quad j \in \mathbb{Z} \, , \quad \text{a.e. } \vartheta \in \mathbb{R} \, ,
\] (1.6.86)
where \( f, h \in l^2(\mathbb{Z}) \). Moreover, for any \( Z \in L^2_{\text{loc}}(\mathbb{Z} \times \mathbb{R}) \) such that
\[
Z(j + L, \vartheta) = e^{2\pi i j \vartheta} Z(j, \vartheta) \, , \quad Z(j, \vartheta + 1) = Z(j, \vartheta) \, , \quad j \in \mathbb{Z} \, , \quad \text{a.e. } \vartheta \in \mathbb{R} \, ,
\] (1.6.87)
there is a unique \( f \in l^2(\mathbb{Z}) \) such that \( Z = Z_L f \). Observe that there is no discrete version of (1.5.6) since the (discrete) Fourier transform does not map \( l^2(\mathbb{Z}) \) into itself.

We shall make the choice \( L = M \), and suppress the subscript \( L \) in \( Z_L \) so that
\[
(Z h)(j, \vartheta) = \sum_{k=-\infty}^{\infty} h(j - kM) e^{2\pi ik\vartheta} \, , \quad j \in \mathbb{Z} \, , \quad \text{a.e. } \vartheta \in \mathbb{R} \, ,
\] (1.6.88)
for $h \in l^2(Z)$. At the end of this subsection we shall indicate how the results given below would look like when the choice $L = N$ instead of $L = M$ were made in $Z_L$.

We set for $f, h \in l^2(Z)$ and $j \in Z$, a.e. $\vartheta \in R$

$$\Phi^f(j, \vartheta) = p^{-1/2} \left( \bigg( Z f \bigg) \left( j - lN, \vartheta + \frac{k}{p} \right) \right)_{k=0, \ldots, p-1; l=0, \ldots, q-1}, \quad (1.6.89)$$

$$A^{fh}(j, \vartheta) = \left( A_{kr}^{fh}(j, \vartheta) \right)_{k,r=0, \ldots, p-1} = \Phi^f(j, \vartheta) \Phi^h(j, \vartheta)^* \quad (1.6.90).$$

Observe that (see (1.6.81) for the definition of $K$)

$$\frac{1}{p} \langle f, h \rangle = \sum_{l=0}^{q-1} \sum_{j=0}^{q-1} \int_0^{p^{-1}} \langle \Phi^f_j(j, \vartheta), \Phi^h_j(j, \vartheta) \rangle \, d\vartheta \quad (1.6.91),$$

where $\Phi^f_j(j, \vartheta), \Phi^h_j(j, \vartheta)$ are the $l$th column of $\Phi^f(j, \vartheta), \Phi^h(j, \vartheta)$, and the inner product at the right-hand side of (1.6.91) is the inner product of $C^p$. Formula (1.6.91) follows from (1.6.83), (1.6.85) and (1.6.86), and the fact that the sets

$$\{lN | l = 0, \ldots, q-1\} \quad \text{and} \quad \{lK | l = 0, \ldots, q-1\} \quad (1.6.92)$$

agree mod $M$ since $\text{GCD}(p, q) = 1$.

1.6.5.1 Frame operator representation

Assume that the system $g_{nN,m/M}, (n, m) \in Z \times I$, has a finite frame upper bound, and let $f \in l^2(Z)$. Then there holds

$$\Phi^{S_f}(j, \vartheta) = A^{gs}(j, \vartheta) \Phi^f(j, \vartheta), \quad j \in Z, \text{ a.e. } \vartheta \in R, \quad (1.6.93)$$

and

$$\langle S_f f, j \rangle = \frac{M}{p} \sum_{l=0}^{q-1} \sum_{j=0}^{q-1} \langle A^{gs} \Phi^f_j(j, \vartheta), \Phi^f_j(j, \vartheta) \rangle \, d\vartheta \quad (1.6.94).$$

These formulas follow in the same way as in the time-continuous case.

1.6.5.2 Frame bounds

For any $A \geq 0, B < \infty$ there holds

$$A \|f\|^2 \leq \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} |(f, g_{nN,m/M})|^2 \leq B \|f\|^2, \quad f \in l^2(Z), \quad (1.6.95)$$

$$\Leftrightarrow \quad \frac{1}{M} A I_{p \times p} \leq A^{gs}(j, \vartheta) \leq \frac{1}{M} B I_{p \times p}, \quad j \in Z, \text{ a.e. } \vartheta \in R. \quad (1.6.96)$$

This follows readily from (1.6.91) and (1.6.94). As in the time-continuous case there holds

$$A^{gs}(j + K, \vartheta) = F A^{gs}(j, \vartheta) F^{-1}, \quad (1.6.97)$$

$$A^{gs}(j, \vartheta + \frac{1}{p}) = J A^{gs}(j, \vartheta) J^{-1}, \quad (1.6.98)$$

with $F$ and $J$ as in Subsec. 1.5.2, so that the right-hand side member of (1.6.95) only needs to be checked for $j = 0, \ldots, K - 1, \vartheta \in [0, p^{-1})$.

1.6.5.3 Fourier series expansion of matrix elements

Let $f, h \in l^2(Z)$. Then $A^{fh}_{kr}(j, \vartheta)$ is $N$-periodic in $j \in Z$ and $1$-periodic in $\vartheta \in R$ with Fourier series

$$A^{fh}_{kr}(j, \vartheta) \sim \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} d_{nmkr} e^{-2\pi inj/N} e^{-2\pi im\vartheta}, \quad (1.6.99)$$

where

$$d_{nmkr} = \begin{cases} \frac{q}{pM} e^{-2\piirm/p} \langle f, h m M, -n/N \rangle & nq \equiv (r - k) \mod p, \\ 0 & \text{otherwise}. \end{cases} \quad (1.6.100)$$

This result can be derived as [Jan95c], Prop. 1.1.

1.6.5.4 Wexler-Raz biorthogonality condition in the Zak transform domain

Assume that $g, \gamma \in l^2(Z)$. Then the two systems $g_{kM,l/N}, (k, l) \in Z \times \{0, \ldots, N - 1\}$, and $\gamma_{kM,l/N}, (k, l) \in Z \times \{0, \ldots, N - 1\}$ are biorthogonal, see (1.6.60), if and only if

$$A^{gs}(j, \vartheta) = \Phi^{\gamma}(j, \vartheta) \Phi^{\gamma}(j, \vartheta)^* = \frac{1}{M} I_{p \times p}, \quad j \in Z, \text{ a.e. } \vartheta \in R. \quad (1.6.101)$$

This follows immediately from 1.6.5.3.

1.6.5.5 Characterization of minimal dual

Assume that the system $g_{nN,m/M}, (n, m) \in Z \times I$, is a frame. Then there holds

$$\left( \Phi^{\gamma \ast}(j, \vartheta) \right)^* = \frac{1}{M} \left( \Phi^{\gamma}(j, \vartheta) \right)^* \left( A^{gs}(j, \vartheta) \right)^{-1}, \quad j \in Z, \text{ a.e. } \vartheta \in R, \quad (1.6.102)$$

and in particular

$$M A^{\gamma \ast}(j, \vartheta) = (M A^{gs}(j, \vartheta))^{-1} \quad (1.6.103)$$

This result follows from 6.5.1 and the fact that $S_{g \ast \gamma} = g$. 

1.6.5.6 Condition A for discrete Weyl-Heisenberg systems

There holds (see Subsec. ref 6.4.4) that

\[ g \in L^2(\mathbb{Z}) \text{ satisfies condition A} \quad (1.6.102) \]

\[ \Leftrightarrow \quad A^{\psi}(j, \vartheta) \text{ has an absolutely convergent Fourier series.} \]

Furthermore, when \( g_{nN,m/M}, \ (n,m) \in \mathbb{Z} \times \mathbb{Z} \), is a frame and \( g \) satisfies condition A, then \( \gamma \gamma \) satisfies condition A as well. These results can be proved in the same way as [Jan95c, Thm. 1.3 and 1.4].

1.6.5.7 Different choice of the Zak transform

We briefly indicate how the results of this subsection look like when we choose \( L = N \), rather than \( M \), in (1.6.82). Denote for \( f, h \in L^2(\mathbb{Z}) \)

\[ \Psi^f(j, \vartheta) = p^{-1/2} \left( (Z_N f) \left( j - kM, \vartheta - \frac{l}{q} \right) \right)_{k=0,...,p-1; l=0,...,q-1} \quad (1.6.103) \]

\[ B^f(h, \vartheta) = \left( B^f_{kr}(j, \vartheta) \right)_{k,r=0,...,p-1} = \Psi^f(j, \vartheta) (\Psi^h(j, \vartheta))^* \quad (1.6.104) \]

Then there holds

\[ \Psi^{Sf}(j, \vartheta) = N B^{\psi}(j, \vartheta) \Psi^f(j, \vartheta), \ j \in \mathbb{Z}, \text{ a.e. } \vartheta \in \mathbb{R}. \quad (1.6.105) \]

Also, the matrix elements \( B^f_{kr}(j, \vartheta) \) are \( N \)-periodic in \( j \in \mathbb{Z} \) and \( q^{-1} \)-periodic in \( \vartheta \in \mathbb{R} \), with Fourier series

\[ B^f_{kr}(j, \vartheta) \sim \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} \epsilon_{nmkr} e^{-2\pi i nj/N} e^{-2\pi i q k \vartheta}, \quad (1.6.106) \]

where

\[ \epsilon_{nmkr} = \frac{q}{pN} e^{2\pi i q k / p} \langle f, h_{(r-k+mp)M, -n/N} \rangle. \quad (1.6.107) \]

From these two results the main results of this subsection can be rederived.