

Exercises “Particle Physics II”

12. The commutation relations of the creation and annihilation operators of a Hermitean scalar field (spin 0 field) with mass m are given by

$$[a(p), a(p')^\dagger] = \underbrace{(2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}')}_{\delta(p,p')} , \quad [a(p), a(p')] = 0 ,$$

where $p^0 = \sqrt{m^2 + \vec{p}^2}$. The one-particle momentum eigenstate $|p\rangle$ is defined by $|p\rangle = a(p)^\dagger |0\rangle$. The general form of a normalizable one-particle state $|\psi^{(1)}\rangle$ is given by

$$|\psi^{(1)}\rangle = \int \underbrace{\frac{d^3p}{(2\pi)^3 2p^0}}_{d\mu(p)} |p\rangle \psi^{(1)}(p) .$$

- (a) $\langle p|p'\rangle = ?$
- (b) Determine the normalization condition for the momentum-space wave function $\psi^{(1)}(p)$ implied by the state normalization $\langle \psi^{(1)} | \psi^{(1)} \rangle = 1$.
- (c) Show that the projection operator

$$P^{(1)} = \int d\mu(p) |p\rangle \langle p|$$

satisfies indeed $P^{(1)} P^{(1)} = P^{(1)}$.

13. The two-particle momentum eigenstate $|p_1, p_2\rangle$ is defined by

$$|p_1, p_2\rangle = a(p_1)^\dagger a(p_2)^\dagger |0\rangle .$$

- (a) $\langle p_1, p_2 | p'_1 p'_2 \rangle = ?$
 - (b) Determine the operator $P^{(2)}$ projecting on the two-particle subspace.
 - (c) Discuss the general form of a normalizable two-particle state $|\psi^{(2)}\rangle$ and the properties of the corresponding two-particle wave function in momentum space.
14. In the case of n particles, one defines

$$|p_1, \dots, p_n\rangle = a(p_1)^\dagger \dots a(p_n)^\dagger |0\rangle .$$

Show by induction:

$$\langle p_1, \dots, p_n | k_1, \dots, k_n \rangle = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n \delta(p_i, k_{\sigma(i)}) .$$

15. Discuss the projection operator $P^{(n)}$ and the general form of an n -particle state $\psi^{(n)}$.

16. The Fourier decomposition of a real scalar field is given by

$$\phi(x) = \int d\mu(p) [a(p)e^{-ipx} + a(p)^\dagger e^{ipx}] .$$

Show that $a(p)$ and $a(p)^\dagger$ can be obtained from $\phi(x)$ by the relations

$$a(p) = i \int d^3x e^{ipx} \overleftrightarrow{\partial}_0 \phi(x) , \quad a(p)^\dagger = -i \int d^3x e^{-ipx} \overleftrightarrow{\partial}_0 \phi(x) .$$

17. Use the previous formulas to show that the canonical equal-time commutation relations for ϕ and $\dot{\phi}$ imply the commutation relations for $a(p)$ and $a(p)^\dagger$ displayed in problem 12.

18. The four-momentum operator P^μ is given by

$$P^\mu = \int d\mu(p) a(p)^\dagger a(p) .$$

Show the following commutation relations:

$$[P^\mu, a(p)] = -p^\mu a(p), \quad [P^\mu, a(p)^\dagger] = p^\mu a(p)^\dagger .$$

19. Show:

$$\exp(iPa)\phi(x)\exp(-iPa) = \phi(x+a) .$$

Hint: It is sufficient to check the infinitesimal version of this relation.

20. Show:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\phi(x-y)\phi(0)|0\rangle .$$

Hint: Use the formula of the previous problem.

21. The propagator of the (free) Klein-Gordon field is defined by

$$\Delta(x) = i\langle 0|T\phi(x)\phi(0)|0\rangle .$$

Show: $\Delta(-x) = \Delta(x)$.

22. Show that $\Delta(x)$ is a Green function of the Klein-Gordon operator, i.e.

$$(\square + m^2)\Delta(x) = \delta^{(4)}(x) .$$

Discuss the behaviour of $\Delta(x)$ for positive (negative) x^0 .