Exercises for nonlocality, entanglement und geometry of quantum systems
Sheet 9

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Exercise 24

The Werner state is given by

$$\rho_W = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{1}{4}\mathbb{1}_4$$

In order to check if the state is entangled for certain p, one has to use a separability criterion. A very ease to calculate criterion is the PPT-criterion (positive partial transposition). Calculate the value of p for which the state becomes separable.

PPT-Criterion
A state ρ acting on $\mathcal{H}^2 \otimes \mathcal{H}^2$ is separable if and only if its partial transposition is a positive operator (all eigenvalues are positive),

$$\rho^{T_B} = (\mathbb{1} \otimes T)\rho \geq 0.$$

It does the following to a matrix:

$$\rho_{ij,kl} = \langle i \otimes k|\rho|j \otimes l\rangle = \begin{pmatrix} \rho_{11,\mathrm{kl}} & \rho_{12,\mathrm{kl}} \\ \rho_{21,\mathrm{kl}} & \rho_{22,\mathrm{kl}} \end{pmatrix}$$

$$\text{PPT} \longrightarrow (\mathbb{1} \otimes T)\rho_{ij,kl} = \rho_{ij,lk} = \begin{pmatrix} \rho_{11,\mathrm{lk}} & \rho_{12,\mathrm{lk}} \\ \rho_{21,\mathrm{lk}} & \rho_{22,\mathrm{lk}} \end{pmatrix}$$

That means the PPT criterion applies the following changes to a density matrix:

$$\rho^{T_B} = \begin{pmatrix} \rho_{11,11} & \rho_{11,12} & \rho_{12,11} & \rho_{12,12} \\ \rho_{11,21} & \rho_{11,22} & \rho_{12,21} & \rho_{12,22} \\ \rho_{21,11} & \rho_{21,12} & \rho_{22,11} & \rho_{22,12} \\ \rho_{21,21} & \rho_{21,22} & \rho_{22,21} & \rho_{22,22} \end{pmatrix}$$

Now one has to check if the eigenvalues of this new density matrix are positive or not.

Exercise 25

Consider the following state:

$$\rho_\alpha = \frac{1}{4}(\mathbb{1} - \alpha\vec{\sigma}\otimes\vec{\sigma})$$

What is the range of α for ρ_α to be a quantum state? What state do you get for $\alpha = 1$? Calculate the nearest separable state ρ_0 (calculate for which α the state becomes separable)?

Exercise 26

The norm of an operator is defined as

$$||A|| = \sqrt{Tr(A^\dagger A)}$$

Calculate

$$||\sigma_A \otimes \sigma_B|| \text{ and the Hilbert-Schmidt distance } D(\rho_\alpha) = ||\rho_0 - \rho_\alpha||$$

ρ_0 is the nearest separable state for ρ_α.

Exercise 27

An entanglement witness is an operator A defined by the following Entanglement Witness Inequalities (EWI):

$$\langle\rho|A\rangle \geq 0 \ \forall\rho \in S \text{ (separable)}$$
$$\langle\rho|A\rangle < 0 \text{ for a certain } \rho \text{ entangled}$$

Calculate the optimal entanglement witness, which is given by

$$A_{max} = \frac{\rho_0 - \rho_\alpha - \langle\rho_0|\rho_0 - \rho_\alpha\rangle\mathbb{1}}{||\rho_0 - \rho_\alpha||}$$

Exercise 28

Check the EWI explicitly for ρ_α and A_{max} (results from Exercises 25 and 27), and use the Bloch decomposition for a general separable state

$$\rho_{sep} = \sum_k p_k \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + n_k^i\sigma^i\otimes\mathbb{1} + m_k^j\mathbb{1}\otimes\sigma^j + n_k^i m_k^j\sigma^i\otimes\sigma^j)$$

$$|\vec{n}_k| \leq 1, \ |\vec{m}_k| \leq 1$$

Exercise 29

The maximal violation of the EWI by an entangled state ρ_{ent} is defined by

$$B(\rho_{ent}) = \max_A \left(\min_{\rho\in S}\langle\rho|A\rangle - \langle\rho_{ent}|A\rangle\right)$$

Check explicitly the BNT Theorem

$$B(\rho_\alpha) = D(\rho_\alpha)$$

for ρ_α with α in the entangled range (Exercise 25).