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Entanglement detection with bounded reference frames

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Abstract. Quantum experiments usually assume the existence of perfect, classical reference frames (RFs), which allow for the specification of measurement settings (e.g. orientation of the Stern-Gerlach magnet in spin measurements) with arbitrary precision. If the RFs are ‘bounded’ (i.e. quantum systems themselves, having a finite number of degrees of freedom), only limited precision can be attained. Using spin coherent states as bounded RFs, we have found the minimum size needed for them to violate local realism for entangled spin systems. For composite systems of spin $\frac{1}{2}$ particles, RFs of very small size are sufficient for the violation; however, to see this violation for macroscopic entangled spins, the size of the RF must be at least quadratically larger than that of the spins. The unavailability of such RFs gives a possible explanation for the non-observance of violation of local realism in everyday experience.

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The Kochen–Specker test of contextuality [1] and Bell's test of local realism [2] provide theory-independent tests of the 'classicality' of a system. In the latter, correlations between space-like separated parts of a composite system are measured for different choices of measurement settings. Certain combinations of these correlations constitute 'test quantities' (Bell's inequalities) that are bounded in all classical (local realistic) theories. Violation of these bounds implies that the tested state has no local realistic explanation. Typically, the choices of the settings correspond to different orientations of the measuring apparatuses (like a polarizer or the Stern–Gerlach magnet). This implicitly assumes the existence of an external (classical) reference frame (RF), which allows one to specify with arbitrary precision the directions chosen. However, what if no RF is available? The impossibility of specifying measurement settings precisely, in the absence of a perfect RF, leads to a kind of 'intrinsic decoherence' [3]–[5] that might wash out all quantum features. *What are the minimum RF resources needed so that the quantum features of a given system can still be observed?*

If we adopt the natural assumption that physical resources in the universe are finite, we will always be confronted with bounded RFs. Physically, this means that our measurements will always be imprecise. It is of fundamental interest to determine the minimum measurement precision required so that one can still observe genuine quantum features, such as contextuality or violation of local realism (see, for example, [6, 7]). In addition to these foundational reasons, the questions given above are relevant for developing methods to contend with bounded RFs using relational encoding, particularly in the context of computation, cryptography, and communication [8]–[13]. For certain tasks, such as quantum key distribution [14] or quantum communication complexity [15], entangled states are useful only to the extent that they violate Bell's inequalities. It is thus important to quantify the costs in RF resources for the violation.

To introduce the idea of measuring relative degrees of freedom in the situation of lacking an external RF, consider the problem of determining the direction towards which a spin $\frac{1}{2}$ particle points. In general the direction can be defined as a relative angle to some macroscopic pointer (e.g. the Stern–Gerlach magnet), which serves as an external RF. Performing many repetitions of the Stern–Gerlach experiment with the same spin state, we can infer this angle. If we are given two spin $\frac{1}{2}$ particles, we can determine the relative angle between them by measuring the angles between each of the spins and the external RF, and from these computing the relative angle. Now suppose that the experimenter has no access to such an external Cartesian RF; operationally, this means that she has no information about her orientation with respect to the rest of the world (but she can still control all the devices in her laboratory). In particular, the angles between her instruments and those used to prepare the particles are not known and may change in every repetition of the experiment. Nonetheless, the relative orientation between the two spins can be measured in a manner which is invariant under rotations. If she measures the

total spin of the two particles, this can take the values 0 and 1. Now we are tempted to say that the two spins add when they are aligned and subtract when they are anti-aligned, and so we can interpret this as the measurement of the projection of the first spin along the direction of the second. This procedure leads unavoidably to errors; e.g. if the spins were initially in the state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle|z-\rangle + |z-\rangle|z+\rangle)$, then a measurement on each particle with respect to an external RF along z would imply that they are anti-aligned along this direction. However, the measurement of total spin is interpreted as if they were aligned, because the total spin of $|\psi^+\rangle$ is 1. As this procedure is proven to be optimal [16], there is a fundamental restriction in the determination of the relative angle between two (finite) spins that the experimenter can achieve in the absence of an external RF. One can apply the same procedure using a spin- j coherent state as the RF; as j becomes larger, the errors introduced decrease and, eventually, for $j \rightarrow \infty$ (unbounded RF) the measurement with the classical RF is exactly reproduced⁶.

In this work, we determine how ‘strong’ RFs need to be to allow violation of Bell’s inequalities. In the Bell test the observers are given bounded RFs. Since we are interested in violations of local realism in the transition from quantum to classical RFs, we use the spin coherent states to represent quantum RFs because they are closest to the notion of a classical direction. We find that a pair of spin $\frac{1}{2}$ particles exhibits violation of Clauser–Horne–Shimony–Holt (CHSH) inequality already for $j_{\text{RF}} > \frac{5}{2}$. In the case of multi-particle Mermin [18] inequalities, even if half of the RFs are of minimal size $j_{\text{RF}} = 1/2$, and the other half ‘unbounded’ (classical), the ratio between the quantum and local realistic bound remains exponential in the number of entangled spins. Finally, for the case of two macroscopic spins exhibiting violations of a Bell inequality when classical RFs are available, we find that the violation is possible with bounded RFs only if their size is *quadratically* larger than the size of the spins. Since our everyday RFs do not meet this requirement for macroscopically large spins, this suggests an explanation as to why we do not see such violations in everyday life. All our results are derived for violations of local realism, but—when applied to the different degrees of freedom of a single system—they can be interpreted as contextuality proofs as well [17].

1. Measurement of relative degrees of freedom

In this work, we consider directional RFs. Given a classical, external RF, one can measure the projection of a spin- j_S particle along a specific direction, where the $2j_S + 1$ possible outcomes $m = -j_S, \dots, j_S$, correspond to the projectors $\Pi_m = |j_S, m\rangle\langle j_S, m|$. Generalizing this, we consider bounded RFs by replacing the classical RF with a quantum RF in the form of a coherent state of spin j_{RF} .

Without an external RF, the experimenter can only measure relational degrees of freedom of the particle and the coherent state $|j_{\text{RF}}\rangle$. The task is thus to estimate the relative angle between them using only rotationally invariant operations. An optimal procedure consists in performing

⁶ To physically implement measurements of total spin, one must be able to entangle the two particles. In practice, information about the alignment of the beams containing the particles with respect to the laboratory would need to be available. However, typically this information about beam alignment does not contain any information about the particle spins—i.e. the internal degrees of freedom of the two particles. Therefore, one can lack RFs for internal degrees of freedom and still be able to perform the required measurements in the lab.

the projective measurement onto the subspaces of total spin j of two spins [8], which can take the values $j = |j_{\text{RF}} - j_S|, \dots, j_{\text{RF}} + j_S$. When the outcome $j = j_{\text{RF}} + m$ of total spin is observed, we associate the spin component m along the direction of the bounded RF to the system. In this way the spin projection measurement relative to a bounded RF simulates the one relative to an unbounded RF. The projectors associated to subspaces of total spin are

$$\hat{\Pi}_{j_{\text{RF}}+m} = \sum_{m'=-j_{\text{RF}}-m}^{j_{\text{RF}}+m} |j_{\text{RF}} + m, m'\rangle \langle j_{\text{RF}} + m, m'|.$$

The effective measurement on the system alone is represented by the POVM elements

$$\hat{P}_{j_S m}^{j_{\text{RF}}} = \langle j_{\text{RF}} | \hat{\Pi}_{j_{\text{RF}}+m} | j_{\text{RF}} \rangle. \quad (1)$$

These can be expressed in terms of the Clebsch–Gordan coefficients $C_{m_1 m_2 m}^{j_1 j_2 j} := \langle j, m | |j_1, m_1\rangle |j_2, m_2\rangle$ and are given by $\hat{P}_{j_S m}^{j_{\text{RF}}} = \sum_{m_S n_S} C_{j_{\text{RF}} m_S j_{\text{RF}}+m}^{j_{\text{RF}} j_S j_{\text{RF}}+m} C_{j_{\text{RF}} n_S j_{\text{RF}}+n_S}^{j_{\text{RF}} j_S j_{\text{RF}}+n_S} |j_S, m_S\rangle \langle j_S, n_S|$. Using the asymptotic properties of the Clebsch–Gordan coefficients [19] it can be shown that $\hat{P}_{j_S m}^{j_{\text{RF}}} \rightarrow |j_S, m\rangle \langle j_S, m| \equiv \hat{\Pi}_m$ when $j_{\text{RF}} \rightarrow \infty$. This shows that the relational measurement, with an increasingly large bounded RF, tends toward the measurement with an unbounded RF.

A more general way to exploit relative degrees of freedom would be to encode information in rotationally invariant subsystems. For example, three spin $\frac{1}{2}$ particles possess a two-level invariant subsystem in the subspace of total spin $\frac{1}{2}$. Thus, one rotationally invariant qubit (quantum bit) can be encoded in three physical ones and six spin $\frac{1}{2}$ particles (three for Alice and three for Bob) would allow perfect violation of Bell’s inequalities with no need for a rotational RF. In this work we do not consider such general schemes. Rather, we study the situations in which ‘system’ and ‘quantum RF’ are separated—resembling the conventional situation in experiments in which ‘system’ and ‘classical RF’ are separated. This will allow us to investigate the quantum features of systems as the bounded quantum RFs approach the classical limit.

In the Bell experiment each of the observers chooses between two or more measurement settings, corresponding, for example, to measurements of spin components along different directions $\vec{\alpha}_1, \vec{\alpha}_2, \dots$. In our scenario this choice corresponds to the use of coherent states $|\vec{\alpha}_1\rangle, |\vec{\alpha}_2\rangle, \dots$ pointing to different directions as RFs. Here $|\vec{\alpha}_i\rangle$ is the eigenstate with the maximal eigenvalue of the spin component along the direction $\vec{\alpha}_i$. One possible way to prepare such states is to apply an appropriate rotation to the given coherent state $|j_{\text{RF}}\rangle$ pointing towards the z -axis. Since, however, Alice and Bob are assumed to have no RF, we consider the following operational realization. In every experimental run a third party (Charlie) sends to both Alice and Bob one coherent state for each possible setting, together with the entangled pair to be measured. Each coherent state is prepared along the direction which would be chosen when classical RFs were used. Having no RF for directions, Alice cannot know the angles of the different coherent states, but she can still distinguish between them (for example, Charlie can send them with a short, agreed, time delay), and can decide which one to use as an RF, allowing the freedom of choice necessary in a Bell experiment.

In an alternative implementation, Charlie chooses the setting and sends only the corresponding (one) coherent state. In this case no additional resources to those actually used are distributed, but the need for freedom of choice for the two observers is not strictly satisfied. One could think of more complicated schemes for ensuring this requirement is met (e.g. introducing more agents that share a global RF with Charlie, but are space-like separated, such that they can perform the settings choice under locality conditions); however, in the present work the focus is

not on a strict disproof of local realistic theories, but rather on the ability to measure quantum correlations under the restriction of bounded RFs.

It is here assumed that the channel between Charlie and the observers is subject to a collective noise; that is to say, all the particles sent to one observer undergo the same unknown rotation (though a different rotation occurs in different runs and for the different observers), and this is important since we want to exploit relative degrees of freedom. Such an assumption could be reasonable in some quantum communication schemes.

2. Violation of Bell's inequalities

As a first example, we consider a Bell experiment on an entangled pair of spin $\frac{1}{2}$ particles with bounded RFs. We will determine the minimum size for the spin RFs where outcomes can still violate Bell's inequalities.

We consider the CHSH inequality [20]

$$S := \left| E(\vec{\alpha}_1, \vec{\beta}_1) + E(\vec{\alpha}_1, \vec{\beta}_2) + E(\vec{\alpha}_2, \vec{\beta}_1) - E(\vec{\alpha}_2, \vec{\beta}_2) \right| \leq 2, \quad (2)$$

where $E(\vec{\alpha}_i, \vec{\beta}_j)$ is the correlation function for the measurement $\vec{\alpha}_i$ at one laboratory and $\vec{\beta}_j$ at the other laboratory. In quantum mechanics, for a given state $\hat{\rho}$ of the pair, if the first spin is measured along direction $\vec{\alpha}_i$ and the second along $\vec{\beta}_j$, the correlation function reads $E(\vec{\alpha}_i, \vec{\beta}_j) = \text{Tr}[\hat{\rho} (\vec{\alpha}_i \cdot \vec{\sigma}) (\vec{\beta}_j \cdot \vec{\sigma})]$.

In contrast to the standard Bell experiment in which two distant observers possess unbounded RFs, we assume that they can only use their coherent states ($|j_1\rangle$ and $|j_2\rangle$) with respect to which entangled spins can be measured. How large must j_1 and j_2 be, so that the CHSH inequality is still violated?

We assume that the pair is in the singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|z+\rangle|z-\rangle - |z-\rangle|z+\rangle)$. The two observers can choose between two measurement settings each, the setting being defined by the direction towards which the RF coherent state is pointing. As $|\psi^-\rangle$ is a rotationally invariant state, the only relevant parameter in the correlation function is the relative angle between the two pointers. It is more mathematically convenient (but operationally equivalent) to write the state with fixed measurement settings, and then to apply a rotation of an angle ϑ to one of the particles. The corresponding rotated singlet state is $|\psi^-(\vartheta)\rangle = \frac{1}{\sqrt{2}}[\sin \frac{\vartheta}{2}(|z+\rangle|z+\rangle + |z-\rangle|z-\rangle) + \cos \frac{\vartheta}{2}(|z+\rangle|z-\rangle - |z-\rangle|z+\rangle)]$. If one measures the total spin of the joint system of particle and RF at the two laboratories, the probabilities of the various outcomes are

$$p_{mn}(\vartheta) = \sum_{m_1 m_2} |\langle j_1 | \langle j_2 | \langle \psi^-(\vartheta) | |j_1 + m, j_1 + m_1\rangle |j_2 + n, j_1 + m_2\rangle|^2,$$

where $m, n = -\frac{1}{2}, \frac{1}{2}$. (i.e. $p_{\frac{1}{2}\frac{1}{2}}(\vartheta)$ is the probability of finding the two particles aligned along $|j_1\rangle$ and $|j_2\rangle$, respectively, $p_{-\frac{1}{2}-\frac{1}{2}}(\vartheta)$ is the probability of finding both anti-aligned, etc). To calculate these one needs the coefficients

$$\begin{aligned} \langle j | \langle z+ | |j + m, j + n\rangle &= \delta_{n, \frac{1}{2}} \delta_{m, \frac{1}{2}}, \\ \langle j | \langle z- | |j + m, j + n\rangle &= \frac{\delta_{n, -\frac{1}{2}}}{\sqrt{2j+1}} \left(\delta_{m, \frac{1}{2}} + \sqrt{2j} \delta_{m, -\frac{1}{2}} \right). \end{aligned} \quad (3)$$

Inserting the probabilities in the definition of the correlation function $E^{j_1 j_2}(\vartheta) = -\sum_{m n = -\frac{1}{2}}^{\frac{1}{2}} (-1)^{m+n} p_{m n}(\vartheta)$ one obtains

$$E^{j_1 j_2}(\vartheta) = \frac{1 - 4j_1 j_2 \cos \vartheta}{(2j_1 + 1)(2j_2 + 1)}. \quad (4)$$

In the limit of large j_1 and j_2 , equation (4) becomes the familiar expression $E(\vartheta) = -\cos \vartheta$ for the singlet correlation function with unbounded RFs. Note that, differing from classical RFs, we have in (4) an offset term before $\cos \vartheta$, so that $E^{j_1 j_2}(\vartheta) \neq E^{j_1 j_2}(\vartheta + \pi)$. This implies that the measurement settings which maximize (2) are for the relative angle $\frac{3}{4}\pi$ in all cases except between $\vec{\alpha}_2$ and $\vec{\beta}_2$, for which the angle is $\frac{\pi}{4}$. The RF-dependent CHSH expression then reads

$$S(j_1, j_2) = 2 \left| \frac{1 + 4\sqrt{2}j_1 j_2}{(2j_1 + 1)(2j_2 + 1)} \right|. \quad (5)$$

It exceeds the local realistic limit of 2 if $j_1 > \frac{j_2}{2(\sqrt{2}-1)j_2-1}$. Therefore, for equal RFs, one needs at least $j_1 = j_2 = \frac{5}{2}$.

3. Mermin inequalities

We explore violation of multi-particle Bell's inequalities with bounded RFs. Consider N spin $\frac{1}{2}$ particles (systems S_1, \dots, S_N), that are measured along directions $\vec{\alpha}_1, \dots, \vec{\alpha}_N$. Each individual measurement can give $\pm\frac{1}{2}$ as a result; a specific outcome is thus labeled by a string $\{\mu_1, \dots, \mu_N\}$, where $\mu_k = 1$ stands for the k th spin detected aligned with $\vec{\alpha}_k$, while $\mu_k = -1$ represents the spin anti-aligned with $\vec{\alpha}_k$. The multiparticle correlation function is defined as $E(\vec{\alpha}_1, \dots, \vec{\alpha}_N) = \sum_{\mu_1, \dots, \mu_N = \pm 1} \prod_{k=1}^N \mu_k p(\mu_1, \dots, \mu_N; \vec{\alpha}_1, \dots, \vec{\alpha}_N)$, where $p(\mu_1, \dots, \mu_N; \vec{\alpha}_1, \dots, \vec{\alpha}_N)$ is the probability for the outcomes μ_1, \dots, μ_N given the settings $\vec{\alpha}_1, \dots, \vec{\alpha}_N$.

The Mermin inequality is given by [18, 21]

$$M := \left| \sum_{x_1, \dots, x_N = 0, 1} \cos \left[\frac{\pi}{2} (x_1 + \dots + x_N) \right] E(\vec{\alpha}_{x_1}, \dots, \vec{\alpha}_{x_N}) \right| \leq 2^{((N-1)/2)}. \quad (6)$$

Using unbounded RFs the Mermin expression reaches its maximal value of $M = 2^{N-1}$ for the Greenberger–Horne–Zeilinger (GHZ) state $|\psi\rangle = \frac{1}{\sqrt{2}} (\otimes_{k=1}^N |z+\rangle_{S_k} + \otimes_{k=1}^N |z-\rangle_{S_k})$ and measurement settings $\vec{\alpha}_0 = X \equiv (\frac{\pi}{2}, 0)$ and $\vec{\alpha}_1 = Y \equiv (\frac{\pi}{2}, \frac{\pi}{2})$ for every particle. Note that the ratio between maximal quantum and local realistic bound increases exponentially with the number of particles: $2^{((N-1)/2)}$.

Again, we assume now that the k th observer, $k = 1, \dots, N$, is given a bounded RF in the form of the coherent state $|j_k\rangle$. Each of the observers measures along directions $\vec{\alpha}_0 \equiv (\vartheta_0, \varphi_0)$ and $\vec{\alpha}_1 \equiv (\vartheta_1, \varphi_1)$. Rewriting the particles in the GHZ state in terms of these directions, we have $|\psi(\vec{\alpha}_1, \dots, \vec{\alpha}_N)\rangle = \frac{1}{\sqrt{2}} (\otimes_{k=1}^N |\vec{\alpha}_k+\rangle_{S_k} + \otimes_{k=1}^N |\vec{\alpha}_k-\rangle_{S_k})$, where $|\vec{\alpha}_k\pm\rangle$ is the state of a particle after the inverse rotation along $\vec{\alpha}_k$ is applied to $|z\pm\rangle$.

As for the two-particle case, in the k th laboratory the total spin of the joint system k th particle + k th RF is measured and the outcome is interpreted as the projection of the particle's

spin along the RF's direction. After a somewhat lengthy but straightforward calculation one obtains the correlation function observed to be

$$E(\vec{\alpha}_1, \dots, \vec{\alpha}_N; j_1, \dots, j_N) = \frac{1}{\prod_{k=1}^N d_k} \left\{ \frac{1}{2} \left[\prod_{k=1}^N (1 + 2j_k \cos \vartheta_k) + \prod_{k=1}^N (1 - 2j_k \cos \vartheta_k) \right] + \cos \left(\sum_{k=1}^N \varphi_k \right) \prod_{k=1}^N 2j_k \sin \vartheta_k \right\}, \quad (7)$$

where $d_k = 2j_k + 1$.

Inserting the correlation function (7) into the left-hand side of the inequality (6), we find for the Mermin expression:

$$M(j_1, \dots, j_k) = \frac{1}{\prod_{k=1}^N d_k} \left| \sqrt{2} \cos(N(\pi/4)) + 2^{N-1} \prod_{k=1}^N (2j_k) \right|. \quad (8)$$

For $j_k \rightarrow \infty$, this approaches the value 2^{N-1} when unbounded RFs are used. In the limit of large number of particles, the Mermin expression becomes

$$M = 2^{N-1} \prod_{k=1}^N \frac{j_k}{j_k + \frac{1}{2}} + \mathcal{O}(1), \quad N \rightarrow \infty. \quad (9)$$

If all the RFs are of the same size, $j_k = j \forall k$, the minimum size of RFs that leads to violation is $j = 3/2$. One can use, however, even fewer resources if one allows spins of different lengths for RFs. If one takes N_1 spins of size j_1 and N_2 of size j_2 , expression (9) becomes $M \approx 2^{N-1} \left(\frac{j_1}{j_1 + \frac{1}{2}} \right)^{N_1} \left(\frac{j_2}{j_2 + \frac{1}{2}} \right)^{N_2}$. In this case the Mermin inequality is violated if

$$\left(\sqrt{2} \frac{j_1}{j_1 + \frac{1}{2}} \right)^{N_1} \left(\sqrt{2} \frac{j_2}{j_2 + \frac{1}{2}} \right)^{N_2} > \sqrt{2}. \quad (10)$$

As both factors are positive, equation (10) can hold only if at least one of the two is larger than 1, which is equivalent to $j_i > \frac{1}{2(\sqrt{2}-1)} = 1.21$. This implies that some of the RFs must have spin size equal to or larger than $\frac{3}{2}$. If the parties have N_1 spin $\frac{3}{2}$ RFs and $N_2 = N - N_1$ spin $\frac{1}{2}$ RFs, the minimum ratio is $\frac{N_1}{N} \cong 0.85$ for seeing violation. Therefore, the minimum resources needed are 85% of spin $\frac{3}{2}$ and 15% of spin $\frac{1}{2}$ RF.

Another interesting case is when a fraction of the RFs is unbounded, which is equivalent to taking the limit $j_2 \rightarrow \infty$ in the inequality (10). For N_1 RFs of size j_1 and N_2 unbounded RFs, it becomes $\left(\sqrt{2} \frac{j_1}{j_1 + \frac{1}{2}} \right)^{N_1} > 2^{((1-N_2)/2)}$. For $j_1 = 1/2$, this is satisfied when $N_2 > N_1 + 1$, which means that when half of the RFs are unbounded and half are as small as spin $\frac{1}{2}$, violation of the Mermin inequality is still possible.

Note that in all cases considered—even when using small quantum RFs—the ratio between the quantum and local realistic bound is still exponential, as can be easily seen by inserting

the results found into the expression (9). However, if a single measurement is replaced with a random guess (corresponding mathematically to $j_1 = 0$), the inequality is satisfied. Thus a non-trivial RF is required for every observer in order to see nonclassicality.

4. Higher spins and the classical limit

It was shown in [6] that violation of Bell's inequalities with entangled systems of arbitrarily large dimension is possible. This shows that the view that large quantum numbers are associated with the classical limit is, in general, erroneous. We will show that to observe violations of local realism for large spins it is necessary to use RFs of sufficient largeness compared to the size of the spins. The scaling of the two sizes is the issue we are interested in.

Following Peres [6], we consider a pair of spin- j_S particles in the generalized singlet state:

$$|\Psi_{j_S}^-\rangle := \frac{1}{\sqrt{2j_S+1}} \sum_{m=-j_S}^{j_S} (-1)^{j_S-m} |j_S, m\rangle |j_S, -m\rangle. \quad (11)$$

And define the parity measurement

$$\hat{P}^c = \sum_{m=-j_S}^{j_S} (-1)^{j_S-m} \hat{\Pi}_m, \quad (12)$$

with $\hat{\Pi}_m = |j_S, m\rangle\langle j_S, m|$ the projectors onto subspaces of the spin component along the z -axis. The parity measurement takes the value $+1$ for all even m , and -1 for all odd m . When the parity operator is defined with respect to spin projection along another direction $\vec{\alpha}$, we will speak about parity measurement $P^c(\vec{\alpha})$ along this direction. For Alice's measurement along the direction $\vec{\alpha}$ and Bob's along $\vec{\beta}$, the correlation function is defined as $E(\vec{\alpha}, \vec{\beta}) = \langle \Psi_{j_S}^- | \hat{P}^c(\vec{\alpha}) \otimes \hat{P}^c(\vec{\beta}) | \Psi_{j_S}^- \rangle$. The CHSH inequality (2) is violated for parity measurements in the singlet state for arbitrarily large spins [6, 22].

To consider violation of the inequality with bounded RFs we introduce a coherent state of length j_{RF} for each observer, and replace the projectors $\hat{\Pi}_m$ in (12) with the POVM from equation (1). When the measurement setting $\vec{\alpha} = (\vartheta, \varphi)$ is chosen, the coherent state $|\vec{\alpha}\rangle$ aligned in that direction is used. In the basis of the spin projection along z -axis, it reads $|\vec{\alpha}\rangle \equiv |\vartheta, \varphi\rangle = \sum_{m=-j_{\text{RF}}}^{j_{\text{RF}}} |m\rangle \binom{2j_{\text{RF}}}{j_{\text{RF}}+m}^{1/2} \cos^{j_{\text{RF}}+m}(\frac{\vartheta}{2}) \sin^{j_{\text{RF}}-m}(\frac{\vartheta}{2}) e^{-im\varphi}$. The rotated POVM is given by $\hat{P}_{j_S m}^{j_{\text{RF}}}(\vec{\alpha}) = \langle \vec{\alpha} | \hat{\Pi}_{j_{\text{RF}}+m} | \vec{\alpha} \rangle$ and the corresponding parity operator by $\hat{P}_{j_S}^{j_{\text{RF}}}(\vec{\alpha}) = \sum_{m=-j_S}^{j_S} (-1)^{j_S-m} \hat{P}_{j_S m}^{j_{\text{RF}}}(\vec{\alpha})$. Finally, the RF-dependent correlation function reads $E_{j_{\text{RF}}}(\vec{\alpha}, \vec{\beta}) = \langle \Psi_{j_S}^- | \hat{P}_{j_S}^{j_{\text{RF}}}(\vec{\alpha}) \otimes \hat{P}_{j_S}^{j_{\text{RF}}}(\vec{\beta}) | \Psi_{j_S}^- \rangle$.

We consider the situation where all the measurement angles are chosen in the same plane ($\varphi = 0$), with the first observer choosing between settings ϑ_1 and ϑ_3 , and the second between ϑ_2 and ϑ_4 . Taking $\vartheta_1 - \vartheta_2 = \vartheta_2 - \vartheta_3 = \vartheta_3 - \vartheta_4 \equiv \Delta\vartheta$, the CHSH inequality reads $S_{j_{\text{RF}}} = |3E_{j_{\text{RF}}}(\Delta\vartheta) - E_{j_{\text{RF}}}(3\Delta\vartheta)| \leq 2$. For classical RFs, the angle difference which maximizes S in the limit of large spin j_S is for $\Delta\vartheta = \frac{x}{2j_S+1}$, with $x = 1.054$. Note that the angle difference is inversely proportional to the spin size. In the case of bounded RFs of finite size j_{RF} it is hard to compute the correlation function analytically, primarily due to the presence of a large number of non-trivial Clebsch–Gordan coefficients. As such we evaluate it, and the CHSH expression,

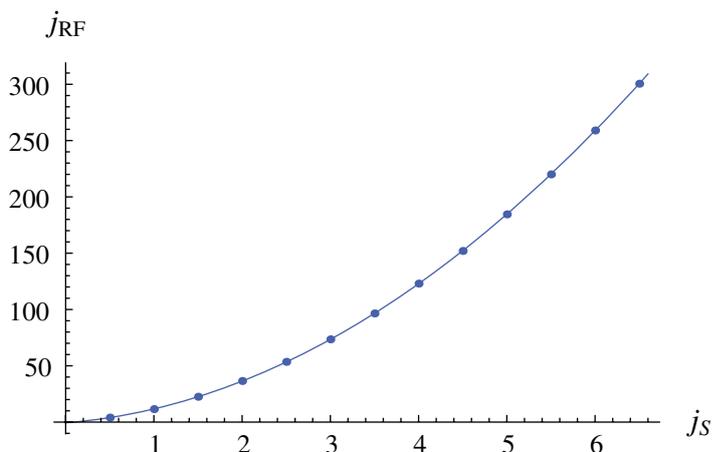


Figure 1. Minimum size of (spin) RF j_{RF} needed for violation of the Bell inequality for measurements on a pair of entangled spins of size j_S . The dots are calculated numerically, the line is the extrapolated fit $j_{RF} \simeq 6j_S^2 + 6j_S$ (see text).

numerically and illustrate the results in figure 1. We see that the size of the RFs needs to scale at least quadratically with the size of entangled spins to observe violation.

We now make a heuristic argument to support our numerical findings. Consider a coherent state of length j_{RF} pointing in a direction $(\vartheta, \varphi = 0)$, and a measurement of spin projection along the z -axis. The probability of obtaining outcome m for the spin z component obeys a binomial distribution $p(m) = \binom{2j_{RF}}{j_{RF}+m} q^{j_{RF}+m} (1-q)^{j_{RF}-m}$, where $q = \cos^2(\frac{\vartheta}{2})$. For large j_{RF} this is approximately a Gaussian centered in $j_{RF} \cos \vartheta$ and with variance $\sigma^2 = \frac{1}{2} j_{RF} \sin^2(\vartheta) \sim j_{RF}$. It can be visualized as an arrow pointing toward ϑ with an angle uncertainty $\frac{1}{\sigma} \sim \frac{1}{\sqrt{2j_{RF}}}$. Using this as a RF it is impossible to distinguish between directions at angles closer than this amount. On the other hand, violation of Bell's inequality requires us to measure setting directions at angles that differ at the order of $\Delta\vartheta \sim \frac{1}{j_S}$. To achieve this precision one needs $\frac{1}{\sigma} < \Delta\vartheta$, which gives the heuristic bound $j_{RF} > j_S^2$. Fitting our numerical results with a quadratic law we indeed find the formula $j_{RF} \simeq 6j_S^2 + 6j_S$ (figure 1). (For higher order fits we obtain coefficients close to zero for powers higher than two).

In conclusion, we have shown how a bounded RF limits the ability of entangled systems to exhibit genuine quantum features, as characterized by violation of Bell's inequalities. This can be relevant in situations where, to implement quantum information tasks, only relational degrees of freedom can be exploited (for example, when the quantum channel is subject to a global noise). We focused on the restrictions derived from the lack of a directional RF, considering that other restrictions would impose additional requirements on the resources needed (see, e.g. [23]).

Quantum behavior is generally not observed in macroscopic systems. One reason is that an extremely high experimental resolution would be required to observe quantum phenomena, larger than can be practically achieved [7]. The quantum nature of any physical (i.e. finite) RF employed in an experiment puts a fundamental limitation on the maximally achievable resolution. Our results set a lower bound on the resources (i.e. the strength of quantum RFs) needed to obtain the resolution that is necessary for observation of the quantum features of a

system with a given size. For example, a small iron magnet can have a magnetic field of around 100 G. Suppose that an entangled state of a pair of spins each with size $j_S \simeq \frac{\hbar 100 G}{\mu_B} \simeq 10^{21}$ (μ_B is the Bohr magneton) were available and could be sufficiently protected from decoherence. Even in this case, according to our analysis, no violation of local realism is possible unless the RFs correspond to magnetic fields at least of order 10^4 G. This is much larger than can generally be found in nature (but still not impossible to produce).

Typically, quantum coherence in a system is quickly lost due to its interaction with the environment, but the coherence is preserved in correlations between the system and the environment [24]. A possible explanation of why we observe the classicality of the macroscopic world rather than these quantum correlations is that they have no operational meaning unless a sufficiently strong RF is available (cf e.g. [25]). But in everyday experience such RFs are not available for systems of the size of the environment. Our analysis suggests that, for directional RFs, at least a quadratic scaling with the size of the system would be required to demonstrate the existence of the quantum correlations.

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