ASC Report No. 16/2015

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www.asc.tuwien.ac.at ISBN 978-3-902627-05-6

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ISBN 978-3-902627-05-6

OPTIMAL CONVERGENCE FOR ADAPTIVE IGA BOUNDARY ELEMENT METHODS FOR WEAKLY-SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. In a recent work [FGHP15], we analyzed a weighted-residual error estimator for isogeometric boundary element methods in 2D and proposed an adaptive algorithm which steers the local mesh-refinement of the underlying partition as well as the multiplicity of the knots. In the present work, we give a mathematical proof that this algorithm leads to convergence even with optimal algebraic rates.

1. INTRODUCTION

1.1. Isogeometric analysis. The central idea of isogeometric analysis (IGA) is to use the same ansatz functions for the discretization of the partial differential equation at hand, as are used for the representation of the problem geometry. Usually, the problem geometry Ω is represented in CAD by means of non-uniform rational B-splines (NURBS), T-splines, or hierarchical splines. This concept, originally invented in [HCB05] for finite element methods (IGAFEM) has proved very fruitful in applications; see also the monograph [CHB09].

Since CAD directly provides a parametrization of the boundary $\partial\Omega$, this makes the boundary element method (BEM) the most attractive numerical scheme, if applicable (i.e., provided that the fundamental solution of the differential operator is explicitly known). However, compared to the IGAFEM literature, only little is found for isogeometric BEM (IGABEM). The latter has first been considered for 2D BEM in [PGK+09] and for 3D BEM in [SSE+13]. Unlike standard BEM with piecewise polynomials which is well-studied in the literature, cf. the monographs [SS11, Ste08] and the references therein, the numerical analysis of IGABEM is widely open. We refer to [SBTR12, PTC13, TM12] for numerical experiments and to [HAD14] for some quadrature analysis. To the best of our knowledge, *a posteriori* error estimation for IGABEM, however, has only been considered for simple 2D model problems in the recent own works [FGP15, FGHP15]. Here, the question is how to adapt the known techniques from standard BEM to non-polynomial ansatz functions and how to exploit the additional freedoms and full potential of IGA.

For standard BEM with discontinuous piecewise polynomials, *a posteriori* error estimation and adaptive mesh-refinement are well understood. We refer to [CMPS04, CMS01, AFF+13] for weighted-residual error estimators and to [FFH+14, FFKP14] for recent overviews on available *a posteriori* error estimation strategies. Moreover, optimal convergence of mesh-refining adaptive algorithms has recently been proved for polyhedral boundaries [FFK+14, FFK+15, FKMP13] as well as smooth boundaries [Gan13]. The work [AFF+15] allows to transfer these results to piecewise smooth boundaries; see also the discussion in the review article [CFPP14]. However, the remarkable flexibility of

Date: June 1, 2015.
Key words and phrases. isogeometric analysis, boundary element method, IGABEM, adaptive algorithm, convergence, optimal convergence rates.

the IGA ansatz functions to manipulate their smoothness properties motivates the development of a new adaptive algorithm which does not only automatically adapt the mesh-width, but also the continuity of the IGA ansatz function.

Adaptive IGAFEM is considered, e.g., in [VGJS11, DJS10]. A rigorous error and convergence analysis in the frame of adaptive IGA is found in [BG15] which proves linear convergence for some adaptive IGAFEM using hierarchical splines for the Poisson equation, while optimal rates are announced for some future work.

1.2. Model problem. We develop and analyze an adaptive algorithm for the following model problem: Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain with $\mathrm{diam}(\Omega) < 1$ and $\Gamma \subseteq \partial\Omega$ be a compact, piecewise smooth part of its boundary with finitely many connected components. We consider the weakly-singular boundary integral equation

$$V\phi(x) := -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,\phi(y)\,dy = f(x) \quad \text{for all } x \in \Gamma, \tag{1.1}$$

where the right-hand side f is given and the density ϕ is sought. To approximate ϕ, we employ a Galerkin boundary element method (BEM) with ansatz spaces consisting of p-th order NURBS. The convergence order for uniform partitions of Γ is usually suboptimal, since the unknown density ϕ may exhibit singularities, which are stronger than the singularities in the geometry. In [FGHP15], we analyzed a weighted-residual error estimator and proposed an adaptive algorithm which uses this *a posteriori* error information to steer the h-refinement of the underlying partition as well as the local smoothness of the NURBS across the nodes of the adaptively refined partitions. It reflects the fact that it is *a priori* unknown, where the singular and smooth parts of the density ϕ are located and where approximation by nonsmooth resp. smooth functions is required. In [FGHP15], we observed experimentally that the proposed algorithm detects singularities and possible jumps of ϕ and leads to optimal convergence behavior. In particular, we observed that the proposed adaptive strategy is also superior to adaptive BEM with discontinuous piecewise polynomials in the sense that our adaptive NURBS discretization requires less degrees of freedom to reach a prescribed accuracy.

1.3. Contributions. We prove that the adaptive algorithm from [FGHP15] is rate optimal in the sense of [CFPP14]: Let μ_ℓ be the weighted-residual error estimator in the ℓ-th step of the adaptive algorithm. First, the adaptive algorithm leads to linear convergence of the error estimator, i.e., $\mu_{\ell+n} \le Cq^n\mu_\ell$ for all $\ell, n \in \mathbb{N}_0$ and some independent constants $C > 0$ and $0 < q < 1$. Moreover, for sufficiently small marking parameters, i.e. agressive adaptive refinement, the estimator decays even with the optimal algebraic convergence rate. Here, the important innovation is that the adaptive algorithm does not only steer the local refinement of the underlying partition (as is the case in previous literature, e.g., [CFPP14, FFK+14, FFK+15, FKMP13, Gan13]), but also the multiplicity of the knots. In particular, the present work is the first available optimality result for adaptive algorithms in the frame of isogeometric analysis. Additionally, we note that we can prove at least plain convergence of the error if the adaptive algorithm is driven by the Faermann estimator η_ℓ analyzed in [FGP15] instead of the weighted-residual estimator μ_ℓ.

Technical contributions of general interest include a novel mesh-size function $h \in L^\infty(\Gamma)$ which is locally equivalent to the element length (i.e., $h|_T \simeq \mathrm{length}(T)$ for all elements T), but also accounts for the knot multiplicity. Moreover, we prove an inverse estimate $\|h^{1/2}\Psi\|_{L^2(\Gamma)} \le C\,\|\Psi\|_{\widetilde{H}^{-1/2}(\Gamma)}$ for NURBS on locally refined meshes. Similar estimates for piecewise polynomials are shown in [DFG+04, GHS05, Geo08].

Throughout, all results apply for piecewise smooth parametrizations γ of Γ and discrete NURBS spaces. In particular, the analysis thus covers the NURBS ansatz used for IGABEM, where the same ansatz functions are used for the discretization of the integral equation and for the resolution of the problem geometry, as well as spline spaces and even piecewise polynomials on the piecewise smooth boundary Γ which can be understood as special NURBS.

1.4. Outline. The remainder of this work is organized as follows: Section 2 fixes the notation and provides the necessary preliminaries. This includes, e.g., the involved Sobolev spaces (Section 2.2), the functional analytic setting of the weakly-singular integral equation (Section 2.3), the assumptions on the parametrization of the boundary Γ (Section 2.4), the discretization of the boundary (Section 2.5), the mesh-refinement strategy (Section 2.6), B-splines and NURBS (Section 2.7), and the IGABEM ansatz spaces (Section 2.8). Section 3 states our adaptive algorithm (Algorithm 3.1) from [FGHP15] and formulates the main theorem on linear convergence with optimal rates (Theorem 3.2) of the weighted-residual estimator μ_ℓ. The linear convergence of μ_ℓ is proved in Section 4. The proof requires an inverse estimate for NURBS in a fractional-order Sobolev norm (Proposition 4.1) as well as a novel mesh-size function for B-spline and NURBS discretizations (Proposition 4.2) which might be of independent interest. The proof of optimal convergence behaviour is given in Section 5. For the empirical verification of the optimal convergence behavior of Algorithm 3.1 and a comparison of IGABEM and standard BEM with discontinuous piecewise polynomials, we refer to the numerous numerical experiments in our preceding work [FGHP15].

2. Preliminaries

2.1. General notation. Throughout, $|\cdot|$ denotes the absolute value of scalars, the Euclidean norm of vectors in $\mathbb{R}^2$, the measure of a set in $\mathbb{R}$ (e.g., the length of an interval), or the arclength of a curve in $\mathbb{R}^2$. The respective meaning will be clear from the context. We write $A \lesssim B$ to abbreviate $A \le cB$ with some generic constant $c > 0$ which is clear from the context. Moreover, $A \simeq B$ abbreviates $A \lesssim B \lesssim A$. Throughout, mesh-related quantities have the same index, e.g., $\mathcal{N}_\star$ is the set of nodes of the partition $\mathcal{T}_\star$, and $h_\star$ is the corresponding local mesh-width etc. The analogous notation is used for partitions $\mathcal{T}_+$ resp. $\mathcal{T}_\ell$ etc.

2.2. Sobolev spaces. For any measurable subset $\Gamma_0 \subseteq \Gamma$, let $L^2(\Gamma_0)$ denote the Lebesgue space of all square integrable functions which is associated with the norm $\|u\|_{L^2(\Gamma_0)}^2 := \int_{\Gamma_0} |u(x)|^2\,dx$. We define the Hilbert space

$$H^{1/2}(\Gamma_0) := \big\{u \in L^2(\Gamma_0) \,:\, \|u\|_{H^{1/2}(\Gamma_0)} < \infty\big\}, \tag{2.1}$$

associated with the Sobolev-Slobodeckij norm

$$\|u\|_{H^{1/2}(\Gamma_0)}^2 := \|u\|_{L^2(\Gamma_0)}^2 + |u|_{H^{1/2}(\Gamma_0)}^2 \quad \text{with} \quad |u|_{H^{1/2}(\Gamma_0)}^2 := \int_{\Gamma_0}\int_{\Gamma_0} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dy\,dx. \tag{2.2}$$

For finite intervals $I \subseteq \mathbb{R}$, we use analogous definitions. By $\widetilde{H}^{-1/2}(\Gamma_0)$, we denote the dual space of $H^{1/2}(\Gamma_0)$, where duality is understood with respect to the extended $L^2(\Gamma_0)$-scalar product, i.e.,

$$\langle u\,;\,\phi\rangle_{\Gamma_0} = \int_{\Gamma_0} u(x)\phi(x)\,dx \quad \text{for all } u \in H^{1/2}(\Gamma_0) \text{ and } \phi \in L^2(\Gamma_0). \tag{2.3}$$

We note that $H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma)$ form a Gelfand triple and all inclusions are dense and compact. Amongst other equivalent definitions of $H^{1/2}(\Gamma_0)$ are the characterization as trace space of functions in $H^1(\Omega)$ as well as equivalent interpolation techniques. All these definitions provide the same space of functions but different norms, where norm equivalence constants depend only on Γ_0; see, e.g., the monographs [HW08, McL00] and the references therein. Throughout our proofs, we shall use the Sobolev-Slobodeckij norm (2.2), since it is numerically computable.

2.3. Weakly-singular integral equation. It is known [HW08, McL00] that the weakly-singular integral operator $V : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ from (1.1) is a symmetric and elliptic isomorphism if $\mathrm{diam}(\Omega) < 1$ which can always be achieved by scaling. For a given right-hand side $f \in H^{1/2}(\Gamma)$, the strong form (1.1) is thus equivalently stated by

$$\langle V\phi\,;\,\psi\rangle_\Gamma = \langle f\,;\,\psi\rangle_\Gamma \quad \text{for all } \psi \in \widetilde{H}^{-1/2}(\Gamma), \tag{2.4}$$

and the left-hand side defines an equivalent scalar product on $\widetilde{H}^{-1/2}(\Gamma)$. In particular, the Lax-Milgram lemma proves existence and uniqueness of the solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)$. Additionally, $V : L^2(\Gamma) \to H^1(\Gamma)$ is well-defined, linear, and continuous.

In the Galerkin boundary element method, the test space $\widetilde{H}^{-1/2}(\Gamma)$ is replaced by some discrete subspace $\mathcal{X}_\star \subset L^2(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma)$. Again, the Lax-Milgram lemma guarantees existence and uniqueness of the solution $\Phi_\star \in \mathcal{X}_\star$ of the discrete variational formulation

$$\langle V\Phi_\star\,;\,\Psi_\star\rangle_\Gamma = \langle f\,;\,\Psi_\star\rangle_\Gamma \quad \text{for all } \Psi_\star \in \mathcal{X}_\star. \tag{2.5}$$

Below, we shall assume that $\mathcal{X}_\star$ is linked to a partition $\mathcal{T}_\star$ of Γ into a set of connected segments.

2.4. Boundary parametrization. Let $\Gamma = \bigcup_i \Gamma_i$ be decomposed into its finitely many connected components Γ_i. Since the Γ_i are compact and piecewise smooth as well, it holds

$$\|u\|_{H^{1/2}(\Gamma)}^2 = \sum_i \|u\|_{H^{1/2}(\Gamma_i)}^2 + \sum_{\substack{i,j\\ i\neq j}} \int_{\Gamma_i}\int_{\Gamma_j} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dy\,dx \simeq \sum_i \|u\|_{H^{1/2}(\Gamma_i)}^2;$$

see, e.g., [FGP15, Section 2.2]. The usual piecewise polynomial and NURBS basis functions have connected support and are hence supported by some *single* Γ_i each. Without loss of generality and for the ease of presentation, we may therefore assume throughout that Γ is connected. All results of this work remain valid for non-connected Γ.

We assume that either $\Gamma = \partial\Omega$ is parametrized by a closed continuous and piecewise two times continuously differentiable path $\gamma : [a,b] \to \Gamma$ such that the restriction $\gamma|_{[a,b)}$ is even bijective, or that $\Gamma \subsetneqq \partial\Omega$ is parametrized by a bijective continuous and piecewise two times continuously differentiable path $\gamma : [a,b] \to \Gamma$. In the first case, we speak of *closed* $\Gamma = \partial\Omega$, whereas the second case is referred to as *open* $\Gamma \subsetneqq \partial\Omega$.

For closed $\Gamma = \partial\Omega$, we denote the $(b-a)$-periodic extension to $\mathbb{R}$ also by γ. For the left and right derivative of γ, we assume that $\gamma'^{\ell}(t) \neq 0$ for $t \in (a,b]$ and $\gamma'^{r}(t) \neq 0$ for $t \in [a,b)$. Moreover we assume that $\gamma'^{\ell}(t) + c\gamma'^{r}(t) \neq 0$ for all $c > 0$ and $t \in [a,b]$ resp. $t \in (a,b)$. Finally, let $\gamma_L : [0,L] \to \Gamma$ denote the arclength parametrization, i.e., $|\gamma_L'^{\ell}(t)| = 1 = |\gamma_L'^{r}(t)|$, and its periodic extension. Elementary differential geometry yields bi-Lipschitz continuity

$$C_\Gamma^{-1} \le \frac{|\gamma_L(s)-\gamma_L(t)|}{|s-t|} \le C_\Gamma \quad \text{for } s,t \in \mathbb{R}, \text{ with } \begin{cases} |s-t| \le \frac{3}{4}L, & \text{for closed } \Gamma,\\ s \neq t \in [0,L], & \text{for open } \Gamma, \end{cases} \tag{2.6}$$

where $C_\Gamma > 0$ depends only on Γ. A proof is given in [Gan14, Lemma 2.1] for closed $\Gamma = \partial\Omega$. For open $\Gamma \subsetneqq \partial\Omega$, the proof is even simpler.

Let $I \subseteq [a,b]$. If $\Gamma = \partial\Omega$ is closed and $|I| \le \frac{3}{4}L$ resp. if $\Gamma \subsetneqq \partial\Omega$ is open, the bi-Lipschitz continuity (2.6) implies

$$C_\Gamma^{-1}|u \circ \gamma_L|_{H^{1/2}(I)} \le |u|_{H^{1/2}(\gamma_L(I))} \le C_\Gamma |u \circ \gamma_L|_{H^{1/2}(I)}. \tag{2.7}$$

2.5. Boundary discretization. In the following, we describe the different quantities which define the discretization.

Nodes $z_j = \gamma(\check{z}_j) \in \mathcal{N}_\star$. Let $\mathcal{N}_\star := \{z_j : j = 1, \dots, n\}$ and $z_0 := z_n$ for closed $\Gamma = \partial\Omega$ resp. $\mathcal{N}_\star := \{z_j : j = 0, \dots, n\}$ for open $\Gamma \subsetneqq \partial\Omega$ be a set of nodes. We suppose that $z_j = \gamma(\check{z}_j)$ for some $\check{z}_j \in [a,b]$ with $a = \check{z}_0 < \check{z}_1 < \check{z}_2 < \dots < \check{z}_n = b$ such that $\gamma|_{[\check{z}_{j-1},\check{z}_j]} \in C^2([\check{z}_{j-1},\check{z}_j])$.

Multiplicity $\#z_j$ and knots $\mathcal{K}_\star, \check{\mathcal{K}}_\star$. Let $p \in \mathbb{N}_0$ be some fixed polynomial order. Each node z_j has a multiplicity $\#z_j \in \{1, 2\dots, p+1\}$ with $\#z_0 = \#z_n = p+1$. This induces knots

$$\mathcal{K}_\star = (\underbrace{z_k, \dots, z_k}_{\#z_k-\text{times}}, \dots, \underbrace{z_n, \dots, z_n}_{\#z_n-\text{times}}), \tag{2.8}$$

with $k = 1$ resp. $k = 0$ and corresponding knots $\check{\mathcal{K}}_\star := \gamma|_{(a,b]}^{-1}(\mathcal{K}_\star)$ resp. $\check{\mathcal{K}}_\star := \gamma^{-1}(\mathcal{K}_\star)$ on the parameter domain $[a,b]$.

Elements, partition $\mathcal{T}_\star$, and $[T]$, $[\mathcal{T}_\star]$. Let $\mathcal{T}_\star = \{T_1, \dots, T_n\}$ be a partition of Γ into compact and connected segments $T_j = \gamma(\check{T}_j)$ with $\check{T}_j = [\check{z}_{j-1}, \check{z}_j]$. We define

$$[\mathcal{T}_\star] := \{[T] : T \in \mathcal{T}_\star\} \quad \text{with} \quad [T] := (T, \#z_{T,1}, \#z_{T,2}), \tag{2.9}$$

where $z_{T,1} = z_{j-1}$ and $z_{T,2} = z_j$ are the two nodes of $T = T_j$.

Local mesh-sizes $h_{\star,T}$, $\check{h}_{\star,T}$ and $h_\star$, $\check{h}_\star$. The arclength of each element $T \in \mathcal{T}_\star$ is denoted by $h_{\star,T}$. We define the local mesh-width function $h_\star \in L^\infty(\Gamma)$ by $h_\star|_T = h_{\star,T}$. Additionally, we define for each element $T \in \mathcal{T}_\star$ its length $\check{h}_{\star,T} := |\gamma^{-1}(T)|$ with respect to the parameter domain $[a,b]$. This gives rise to $\check{h}_\star \in L^\infty(\Gamma)$ with $\check{h}_\star|_T = \check{h}_{\star,T}$. Note that the lengths $h_{\star,T}$ and $\check{h}_{\star,T}$ of an element T are equivalent, where the equivalence constants depend only on γ.

Local mesh-ratios $\check{\kappa}_\star$. We define the local mesh-ratio by

$$\check{\kappa}_\star := \max\{\check{h}_{\star,T}/\check{h}_{\star,T'} : T, T' \in \mathcal{T}_\star \text{ with } T \cap T' \neq \emptyset\}. \tag{2.10}$$

Patches $\omega_\star(z)$, $\omega_\star(U)$, $\omega_\star(\mathcal{U})$, and $\bigcup\mathcal{U}$. For each set $U \subseteq \Gamma$, we inductively define for $m \in \mathbb{N}_0$

$$\omega_\star^m(U) := \begin{cases} U & \text{if } m = 0, \\ \omega_\star(U) := \bigcup\{T \in \mathcal{T}_\star : T \cap U \neq \emptyset\} & \text{if } m = 1, \\ \omega_\star(\omega_\star^{m-1}(U)) & \text{if } m > 1. \end{cases}$$

For nodes $z \in \Gamma$, we abbreviate $\omega_\star(z) =: \omega_\star(\{z\})$. For each set $\mathcal{U} \subseteq [\mathcal{T}_\star]$, we define

$$\bigcup\mathcal{U} := \bigcup\{T \in \mathcal{T}_\star : [T] \in \mathcal{U}\},$$

and

$$\omega_\star^m(\mathcal{U}) := \omega_\star^m(\bigcup\mathcal{U}).$$

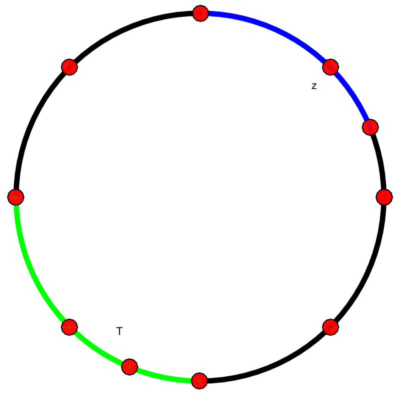

FIGURE 2.1. The patch $\omega_\star(z)$ of some node $z \in \mathcal{N}_\star$ resp. the patch $\omega_\star(T)$ are illustrated in blue resp. green.

2.6. Mesh-refinement. Suppose that we are given a deterministic mesh-refinement strategy $\texttt{ref}(\cdot)$ such that, for each mesh $[\mathcal{T}_\star]$ and an arbitrary set of marked nodes $\mathcal{M}_\star \subseteq \mathcal{N}_\star$, the application $[\mathcal{T}_+] := \texttt{ref}([\mathcal{T}_\star], \mathcal{M}_\star)$ provides a mesh in the sense of Section 2.5 such that, first, the marked nodes belong to the union of the refined elements, i.e., $\mathcal{M}_\star \subset \bigcup([\mathcal{T}_\star] \setminus [\mathcal{T}_+])$, and, second, the knots $\mathcal{K}_\star$ form a subsequence of the knots $\mathcal{K}_+$. The latter implies the estimate

$$|[\mathcal{T}_\star] \setminus [\mathcal{T}_+]| \leq 2(|\mathcal{K}_+| - |\mathcal{K}_\star|), \tag{2.11}$$

since $[\mathcal{T}_\star] \setminus [\mathcal{T}_+]$ is the set of all elements in which a new knot is inserted and one new knot can be inserted in at most 2 elements of the old mesh, i.e., at the intersection of 2 elements.

We write $[\mathcal{T}_+] \in \texttt{ref}([\mathcal{T}_\star])$, if there exist finitely many meshes $[\mathcal{T}_1], \dots, [\mathcal{T}_\ell]$ and subsets $\mathcal{M}_j \subseteq \mathcal{N}_j$ of the corresponding nodes such that $[\mathcal{T}_\star] = [\mathcal{T}_1]$, $[\mathcal{T}_+] = [\mathcal{T}_\ell]$, and $[\mathcal{T}_j] = \texttt{ref}([\mathcal{T}_{j-1}], \mathcal{M}_{j-1})$ for all $j = 2, \dots, \ell$, where we formally allow $m = 1$, i.e., $[\mathcal{T}_\star] = [\mathcal{T}_1] \in \texttt{ref}([\mathcal{T}_\star])$.

For the proof of our main result, we need the following assumptions on $\texttt{ref}(\cdot)$.

Assumption 2.1. *For an arbitrary initial mesh $[\mathcal{T}_0]$ and $[\mathbb{T}] := \texttt{ref}([\mathcal{T}_0])$, we assume that the mesh-refinement strategy satisfies the properties* (M1)–(M3)*:*

(M1) *There exists a constant $\check{\kappa}_{\max} \geq 1$ such that the local mesh-ratios* (2.10) *are uniformly bounded*

$$\check{\kappa}_\star \leq \check{\kappa}_{\max} \quad \textit{for all } [\mathcal{T}_\star] \in [\mathbb{T}]. \tag{2.12}$$

(M2) *For all $[\mathcal{T}_\star], [\mathcal{T}_+] \in [\mathbb{T}]$, there is a common refinement $[\mathcal{T}_\star \oplus \mathcal{T}_+] \in \texttt{ref}([\mathcal{T}_\star]) \cap \texttt{ref}([\mathcal{T}_+])$ such that the knots $\mathcal{K}_\star \oplus \mathcal{K}_+$ of $[\mathcal{T}_\star \oplus \mathcal{T}_+]$ satisfy the overlay estimate*

$$|\mathcal{K}_\star \oplus \mathcal{K}_+| \leq |\mathcal{K}_\star| + |\mathcal{K}_+| - |\mathcal{K}_0|. \tag{2.13}$$

(M3) *Each sequence $[\mathcal{T}_\ell] \in [\mathbb{T}]$ of meshes generated by successive mesh-refinement, i.e., $[\mathcal{T}_j] = \texttt{ref}([\mathcal{T}_{j-1}], \mathcal{M}_{j-1})$ for all $j \in \mathbb{N}$ and arbitrary $\mathcal{M}_j \subseteq \mathcal{N}_j$, satisfies*

$$|\mathcal{K}_\ell| - |\mathcal{K}_0| \leq C_{\mathrm{mesh}} \sum_{j=0}^{\ell-1} |\mathcal{M}_j| \quad \textit{for } \ell \in \mathbb{N}, \tag{2.14}$$

where $C_{\mathrm{mesh}} > 0$ depends only on $[\mathcal{T}_0]$.

This assumption is especially satisfied for the concrete strategy used in [FGP15] and [FGHP15]. This strategy looks as follows: Let $[\mathcal{T}_\star] \in [\mathbb{T}]$. Let $\mathcal{M}_\star \subseteq \mathcal{N}_\star$ be a set of marked nodes. To get the refined mesh $[\mathcal{T}_+] := \mathtt{ref}([\mathcal{T}_\star], \mathcal{M}_\star)$, we proceed as follows:

(i) If both nodes of an element $T \in \mathcal{T}_\star$ belong to $\mathcal{M}_\star$, T will be marked.
(ii) For all other nodes in $\mathcal{M}_\star$, the multiplicity will be increased if it is less or equal to $p+1$, otherwise the elements which contain one of these nodes $z \in \mathcal{M}_\star$, will be marked.
(iii) Recursively, mark further elements $T' \in \mathcal{T}_\star$ for refinement if there exists a marked element $T \in \mathcal{T}_\star$ with $T \cap T' \neq \emptyset$ and $\check{h}_{\star,T'} > \check{\kappa}_0 \check{h}_{\star,T}$.
(iv) Refine all marked elements $T \in \mathcal{T}_\star$ by bisection and hence obtain $[\mathcal{T}_+]$.

According to [AFF+13], the proposed recursion in step (iii) terminates and the generated partition $\mathcal{T}_+$ guarantees (M1) with $\check{\kappa}_{\max} = 2\check{\kappa}_0$. The following proposition shows that also the assumptions (M2)–(M3) are satisfied.

Proposition 2.2. *The proposed refinement strategy* $\mathtt{ref}(\cdot)$ *used in* [FGP15, FGHP15] *satisfies Assumption 2.1, where* $\check{\kappa}_{\max} = 2\check{\kappa}_0$ *and* C_{mesh} *depends only on the initial partition of the parameter domain, i.e.,* $\mathcal{T}_0$ *transformed onto* $[a,b]$.

Proof. For any partition $\mathcal{T}_\star$ of Γ and any subset of marked elements $\mathcal{S}_\star \subseteq \mathcal{T}_\star$, let $\widetilde{\mathtt{ref}}(\mathcal{T}_\star, \mathcal{S}_\star)$ be the partition obtained from the recursive bisection in step (iii)–(iv) above. This local h-refinement procedure has been analyzed in [AFF+13]. According to [AFF+13, Theorem 2.3], the recursion is well-defined and guarantees $\check{\kappa}_\star \le 2\check{\kappa}_0$ for all $\mathcal{T}_\star \in \widetilde{\mathtt{ref}}(\mathcal{T}_0)$.

To see (M2), [AFF+13, Theorem 2.3] guarantees the existence of some coarsest common refinement $\mathcal{T}_\star \widetilde{\oplus} \mathcal{T}_+ \in \widetilde{\mathtt{ref}}(\mathcal{T}_\star) \cap \widetilde{\mathtt{ref}}(\mathcal{T}_+)$ such that

$$|\mathcal{T}_\star \widetilde{\oplus} \mathcal{T}_+| \le |\mathcal{T}_\star| + |\mathcal{T}_+| - |\mathcal{T}_0|.$$

The corresponding nodes just satisfy $\mathcal{N}_\star \oplus \mathcal{N}_+ = \mathcal{N}_\star \cup \mathcal{N}_+$. There exists a finite sequence of meshes $\mathcal{T}_\star = \widetilde{\mathcal{T}}_1, \widetilde{\mathcal{T}}_2 = \widetilde{\mathtt{ref}}(\widetilde{\mathcal{T}}_1, \mathcal{S}_1), \dots, \widetilde{\mathcal{T}}_\ell = \widetilde{\mathtt{ref}}(\widetilde{\mathcal{T}}_{\ell-1}, \mathcal{S}_{\ell-1}) = \mathcal{T}_\star \widetilde{\oplus} \mathcal{T}_+$ with suitable $\mathcal{S}_j \subseteq \mathcal{T}_j$ for $j = 1, \dots, \ell-1$. If we define $\mathcal{M}_j \subseteq \mathcal{N}_j$ as the set of all nodes in $\mathcal{S}_j$, we see that the sequence $[\mathcal{T}_\star] = [\mathcal{T}_1], [\mathcal{T}_2] = \mathtt{ref}([\mathcal{T}_1], \mathcal{M}_1), \dots [\mathcal{T}_\ell] = \mathtt{ref}([\mathcal{T}_{\ell-1}, \mathcal{M}_{\ell-1})$ satisfies $\mathcal{T}_j = \widetilde{\mathcal{T}}_j$ for $j = 1, \dots \ell$. By repetitively marking one single node, we obtain from $[\mathcal{T}_\ell]$ a mesh $[\mathcal{T}_\star \oplus \mathcal{T}_+]$ with nodes $\mathcal{N}_\star \oplus \mathcal{N}_+ = \mathcal{N}_\star \cup \mathcal{N}_+$ and $\#z = \max(\#_\star z, \#_+ z)$, where $\#_\star$ resp. $\#_+$ denote the multiplicity in $\mathcal{K}_\star$ resp. $\mathcal{K}_+$ and, e.g., $\#_+ z := 0$ if $z \in \mathcal{N}_\star \setminus \mathcal{N}_+$. There obviously holds

$$|\mathcal{K}_\star \oplus \mathcal{K}_+| = \sum_{z \in \mathcal{N}_\star \cup \mathcal{N}_+} \#z \le |\mathcal{K}_\star| + |\mathcal{K}_+| - |\mathcal{K}_0|.$$

Moreover, $[\mathcal{T}_\star \oplus \mathcal{T}_+]$ is clearly a refinement of $[\mathcal{T}_+]$ as well.

Finally we consider (M3). As before we have $\mathcal{T}_1 = \widetilde{\mathtt{ref}}(\mathcal{T}_0, \mathcal{S}_0), \dots, \mathcal{T}_\ell = \widetilde{\mathtt{ref}}(\mathcal{T}_{\ell-1}, \mathcal{S}_{\ell-1})$ for suitable $\mathcal{S}_j \subseteq \mathcal{T}_j$, $j = 0, \dots, \ell-1$. Note that there holds $|\mathcal{S}_j| \le 2|\mathcal{M}_j|$. We denote $|\#_j| := |\mathcal{K}_{j+1}| - |\mathcal{K}_j| - (|\mathcal{N}_{j+1}| - |\mathcal{N}_j|)$ as the number of multiplicity increases during the j-th refinement. There holds

$$|\mathcal{K}_{j+1}| - |\mathcal{K}_j| = |\mathcal{T}_{j+1}| - |\mathcal{T}_j| + |\#_j|$$

and hence

$$|\mathcal{K}_\ell| - |\mathcal{K}_0| = |\mathcal{T}_\ell| - |\mathcal{T}_0| + \sum_{j=0}^{\ell-1} |\#_j|.$$

The term $|\mathcal{T}_\ell| - |\mathcal{T}_0|$ can be estimated by $C \sum_{j=0}^{\ell-1} |\mathcal{S}_j|$ with some constant $C > 0$ which depends only on the initial partition of the parameter domain, see [AFF+13, Theorem 2.3], and hence by $2C\sum_{j=0}^{\ell-1}|\mathcal{M}_j|$. The estimate $|\#_j| \le |\mathcal{M}_j|$ concludes the proof with $C_{\rm mesh} = 2C+1$. $\square$

2.7. B-splines and NURBS. Throughout this subsection, we consider *knots* $\check{\mathcal{K}} := (t_i)_{i\in\mathbb{Z}}$ on $\mathbb{R}$ with multiplicity $\#t_i$ which satisfy $t_{i-1} \le t_i$ for $i \in \mathbb{Z}$ and $\lim_{i\to\pm\infty} t_i = \pm\infty$. Let $\check{\mathcal{N}} := \{t_i \,:\, i\in\mathbb{Z}\} = \{\check{z}_j \,:\, j \in \mathbb{Z}\}$ denote the corresponding set of nodes with $\check{z}_{j-1} < \check{z}_j$ for $j\in\mathbb{Z}$. For $i\in\mathbb{Z}$, the i-th *B-spline* of degree p is defined inductively by

$$\begin{aligned} &B_{i,0} := \chi_{[t_{i-1},t_i)},\\ &B_{i,p} := \beta_{i-1,p}B_{i,p-1} + (1-\beta_{i,p})B_{i+1,p-1} \quad \text{for } p\in\mathbb{N}, \end{aligned} \tag{2.15}$$

where, for $t\in\mathbb{R}$,

$$\beta_{i,p}(t) := \begin{cases} \frac{t-t_i}{t_{i+p}-t_i} & \text{if } t_i \ne t_{i+p},\\ 0 & \text{if } t_i = t_{i+p}. \end{cases}$$

We also use the notations $B_{i,p}^{\check{\mathcal{K}}} := B_{i,p}$ and $\beta_{i,p}^{\check{\mathcal{K}}} := \beta_{i,p}$ to stress the dependence on the knots $\check{\mathcal{K}}$. The following lemma collects some basic properties of B-splines.

Lemma 2.3. *Let $I = [a,b)$ be a finite interval and $p\in\mathbb{N}_0$. Then, the following assertions* (i)–(vi) *hold:*

- (i) *The set $\{B_{i,p}|_I \,:\, i\in\mathbb{Z}, B_{i,p}|_I \ne 0\}$ is a basis for the space of all right-continuous $\check{\mathcal{N}}$-piecewise polynomials of degree lower or equal p on I which are, at each knot t_i, $p-\#t_i$ times continuously differentiable if $p-\#t_i \ge 0$.*
- (ii) *For $i\in\mathbb{Z}$, $B_{i,p}$ vanishes outside the interval $[t_{i-1}, t_{i+p})$. It is positive on the open interval (t_{i-1}, t_{i+p}).*
- (iii) *For $i\in\mathbb{Z}$, $B_{i,p}$ is completely determined by the $p+2$ knots $t_{i-1},\dots,t_{i+p}$.*
- (iv) *The B-splines of degree p form a (locally finite) partition of unity, i.e.,*

$$\sum_{i\in\mathbb{Z}} B_{i,p} = 1 \quad \text{on } \mathbb{R}. \tag{2.16}$$

- (v) *For $i\in\mathbb{Z}$ and $s\in\mathbb{R}$, we have*

$$\forall t\in\mathbb{R}: \quad B_{i,p}^{s+\check{\mathcal{K}}}(t) = B_{i,p}^{\check{\mathcal{K}}}(t-s), \tag{2.17}$$

and for $c>0$

$$\forall t\in\mathbb{R}: \quad B_{i,p}^{c\check{\mathcal{K}}}(t) = B_{i,p}^{\check{\mathcal{K}}}(t/c). \tag{2.18}$$

- (vi) *For $\ell\in\mathbb{N}$, let $\check{\mathcal{K}}_\ell = (t_{\ell,i})_{i\in\mathbb{Z}}$ be a sequence of knots such that $\#t_{i,\ell} = \#t_i$ for all $i\in\mathbb{Z}$. If $(\check{\mathcal{K}}_\ell)_{\ell\in\mathbb{N}}$ converges pointwise to $\check{\mathcal{K}}$, then $\big(B_{i,p}^{\check{\mathcal{K}}_\ell}\big)_{\ell\in\mathbb{N}}$ converges almost everywhere to $B_{i,p}^{\check{\mathcal{K}}}$ for all $i\in\mathbb{N}$.*

Proof. The proof of (i) is found in [dB86, Theorem 6], and (ii)–(iii) are proved in [dB86, Section 2]. (iv) is proved in [dB86, page 9–10]. Finally, (v)–(vi) is proved in [FGP15, Lemma 4.2]. $\square$

In addition to the knots $\check{\mathcal{K}} = (t_i)_{i\in\mathbb{Z}}$, we consider positive weights $\mathcal{W} := (w_i)_{i\in\mathbb{Z}}$ with $w_i > 0$. For $i\in\mathbb{Z}$ and $p\in\mathbb{N}_0$, we define the i-th NURBS by

$$R_{i,p} := \frac{w_i B_{i,p}}{\sum_{\ell\in\mathbb{Z}} w_\ell B_{\ell,p}}. \tag{2.19}$$

We also use the notation $R_{i,p}^{\check{\mathcal{K}},\mathcal{W}} := R_{i,p}$. Note that the denominator is locally finite and positive.

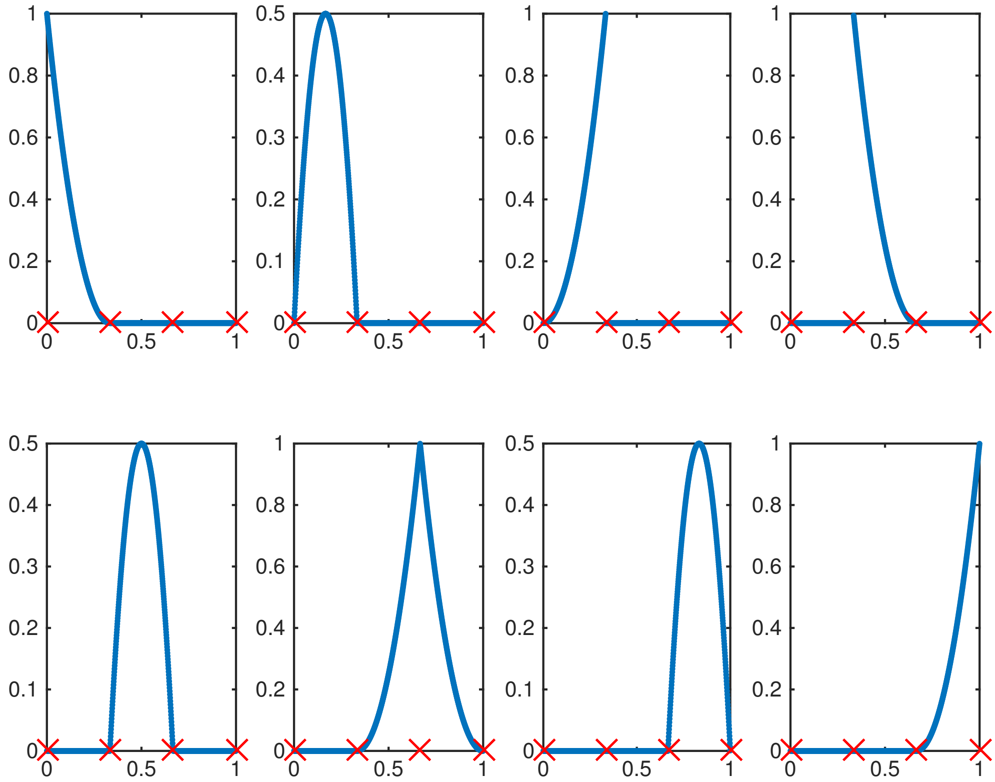

FIGURE 2.2. B-splines on the interval $[0,1]$ corresponding to knot sequence $(\dots,0,0,0,1/3,1/3,1/3,2/3,2/3,1,1,1,\dots)$.

For any $p \in \mathbb{N}_0$, we define the B-spline space

$$\mathscr{S}^p(\check{\mathcal{K}}) := \left\{ \sum_{i\in\mathbb{Z}} a_i B_{i,p} : a_i \in \mathbb{R} \right\} \tag{2.20}$$

as well as the NURBS space

$$\mathscr{N}^p(\check{\mathcal{K}},\mathcal{W}) := \left\{ \sum_{i\in\mathbb{Z}} a_i R_{i,p} : a_i \in \mathbb{R} \right\} = \frac{\mathscr{S}^p(\check{\mathcal{K}})}{\sum_{i\in\mathbb{Z}} w_\ell B_{\ell,p}^{\check{\mathcal{K}}}}. \tag{2.21}$$

2.8. Ansatz spaces. Let $[\mathcal{T}_0]$ be a given initial mesh with corresponding knots $\mathcal{K}_0$ such that $h_0 \le |\Gamma|/4$ for closed $\Gamma = \partial\Omega$. We set $[\mathbb{T}] := \texttt{ref}([\mathcal{T}_0])$. Suppose that $\mathcal{W}_0 = (w_i)_{i=1}^N$ are given initial weights with $N = |\mathcal{K}_0|$ for closed $\Gamma = \partial\Omega$ resp. $N = |\mathcal{K}_0| - (p+1)$ for open $\Gamma \subsetneqq \partial\Omega$.

If $\Gamma = \partial\Omega$ is closed, we extend the transformed knot sequence $\check{\mathcal{K}}_0 = (t_i)_{i=1}^N$ arbitrarily to $(t_i)_{i\in\mathbb{Z}}$ with $t_{-p} = \dots = t_0 = a$, $t_i \le t_{i+1}$, $\lim_{i\to\pm\infty} t_i = \pm\infty$ and $\mathcal{W}_0 = (w_i)_{i\in\mathbb{Z}}$ with $w_i > 0$. For the extended sequences, we also write $\check{\mathcal{K}}_0$ and $\mathcal{W}_0$ and set

$$\mathcal{X}_0 := \mathscr{N}^p(\check{\mathcal{K}}_0,\mathcal{W}_0)|_{[a,b)} \circ \gamma|_{[a,b)}^{-1}. \tag{2.22}$$

If $\Gamma \subsetneqq \partial\Omega$ is open, we extend the sequences $\check{\mathcal{K}}_0 = (t_i)_{i=-p}^N$ and $\mathcal{W}_0$ arbitrarily to $(t_i)_{i\in\mathbb{Z}}$ with $t_i \le t_{i+1}$, $\lim_{i\to\pm\infty} t_i = \pm\infty$ and $\mathcal{W}_0 = (w_i)_{i\in\mathbb{Z}}$ with $w_i > 0$. This allows to define

$$\mathcal{X}_0 := \mathscr{N}^p(\check{\mathcal{K}}_0, \mathcal{W}_0)|_{[a,b]} \circ \gamma^{-1}. \tag{2.23}$$

Due to Lemma 2.3, this definition does not depend on how the sequences are extended.

Let $[\mathcal{T}_\star] \in [\mathbb{T}]$ be a mesh with knots $\mathcal{K}_\star$. Via *knot insertion* from $\mathcal{K}_0$ to $\mathcal{K}_\star$, one obtains unique corresponding weights $\mathcal{W}_\star$. These are chosen such that the denominators of the NURBS functions do not change. In particular, this implies nestedness

$$\mathcal{X}_\star \subseteq \mathcal{X}_+ \quad \text{for all } [\mathcal{T}_\star] \in [\mathbb{T}], [\mathcal{T}_+] \in \mathtt{ref}(\mathcal{T}_\star), \tag{2.24}$$

where the spaces $\mathcal{X}_\star$ resp. $\mathcal{X}_+$ are defined analogously to (2.22)–(2.23). Moreover, the weights are just convex combinations of $\mathcal{W}_0$, wherefore

$$w_{\min} := \min(\mathcal{W}_0) \le \min(\mathcal{W}_\star) \le \max(\mathcal{W}_\star) \le \max(\mathcal{W}_0) =: w_{\max}. \tag{2.25}$$

For further details, we refer to, e.g., [FGP15, Section 4.2].

3. Adaptive algorithm and main results

For each mesh $[\mathcal{T}_\star] \in [\mathbb{T}]$, define the node-based error estimator

$$\mu_\star^2 = \sum_{z\in\mathcal{N}_\star} \mu_\star(z)^2, \tag{3.1a}$$

where the refinement indicators read

$$\mu_\star(z)^2 := |\gamma^{-1}(\omega_\star(z))| \|\partial_\Gamma(f - V\Phi_\star)\|^2_{L^2(\omega_\star(z))} \quad \text{for all } z \in \mathcal{N}_\star. \tag{3.1b}$$

It has been proved in [FGHP15] that $\mu_\star$ is reliable, i.e.,

$$\|\phi - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} \le C_{\rm rel}\, \mu_\star, \tag{3.2}$$

where $C_{\rm rel} > 0$ depends only on p, $w_{\min}$, $w_{\max}$, γ, and $\check{\kappa}_{\max}$. We note that the weighted-residual error estimator in the form $\mu_\star \simeq \|h_\star^{1/2}\partial_\Gamma(f - V\Phi_\star)\|_{L^2(\Gamma)}$ goes back to the works [CS96, Car97], where reliability (3.2) is proved for standard 2D BEM with piecewise constants on polyhedral geometries, while the corresponding result for 3D BEM is found in [CMS01]. We consider the following adaptive algorithm which employs the Dörfler marking strategy (3.3) from [Dör96] to single out nodes for refinement.

Algorithm 3.1. ***Input:*** *Adaptivity parameter* $0 < \theta < 1$, $C_{\rm mark} \ge 1$, *polynomial order* $p \in \mathbb{N}_0$, *initial mesh* $[\mathcal{T}_0]$, *initial weights* $\mathcal{W}_0$.
Adaptive loop: *For each* $\ell = 0, 1, 2, \dots$ *iterate the following steps* (i)–(iv)*:*

(i) *Compute discrete approximation* $\Phi_\ell \in \mathcal{X}_\ell$ *from Galerkin BEM.*
(ii) *Compute refinement indicators* $\mu_\ell(z)$ *for all nodes* $z \in \mathcal{N}_\ell$.
(iii) *Determine an up to the multiplicative constant* $C_{\rm mark}$ *minimal set of nodes* $\mathcal{M}_\ell \subseteq \mathcal{N}_\ell$ *such that*

$$\theta\, \mu_\ell^2 \le \sum_{z\in\mathcal{M}_\ell} \mu_\ell(z)^2. \tag{3.3}$$

(iv) *Generate refined mesh* $[\mathcal{T}_{\ell+1}] := \mathtt{ref}([\mathcal{T}_\ell], \mathcal{M}_\ell)$.

Output: *Approximate solutions* Φ_ℓ *and error estimators* μ_ℓ *for all* $\ell \in \mathbb{N}_0$.

Our main result is that the proposed algorithm is linearly convergent, even with the optimal algebraic rate. For a precise statement of this assertion, let $[\mathbb{T}_N] := \big\{[\mathcal{T}_\star] \in [\mathbb{T}] : |\mathcal{K}_\star| - |\mathcal{K}_0| \le N\big\}$ be the finite set of all refinements having at most N knots more than $[\mathcal{T}_0]$. Following [CFPP14], we introduce an estimator-based approximation class $\mathbb{A}_s$ for $s > 0$: We write $\phi \in \mathbb{A}_s$ if

$$\|\phi\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}_0} \Big((N+1)^s \min_{[\mathcal{T}_\star]\in[\mathbb{T}_N]} \mu_\star\Big) < \infty. \tag{3.4}$$

In explicit terms, this just means that an algebraic convergence rate of $\mathcal{O}(N^{-s})$ for the estimator is possible, if the optimal meshes are chosen.

Theorem 3.2. *Let $f \in H^1(\Gamma)$, so that the weighted-residual error estimator* (3.1) *is well-defined. We suppose that the Assumption 2.1 on the mesh-refinement holds true. Then, for each $0 < \theta \le 1$, there exist constants $0 < q_{\rm lin} < 1$ and $C_{\rm lin} > 0$ such that Algorithm 3.1 is linearly convergent in the sense of*

$$\mu_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n\, \mu_\ell \quad \textit{for all } \ell, n \in \mathbb{N}_0. \tag{3.5}$$

In particular, this implies convergence

$$C_{\rm rel}^{-1}\, \|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \le \mu_\ell \le C_{\rm lin} q_{\rm lin}^\ell\, \mu_0 \xrightarrow{\ell\to\infty} 0. \tag{3.6}$$

Moreover, there is a constant $0 < \theta_{\rm opt} < 1$ such that for all $0 < \theta < \theta_{\rm opt}$, there exists a constant $C_{\rm opt} > 0$ such that for all $s > 0$, it holds

$$\phi \in \mathbb{A}_s \quad \Longleftrightarrow \quad \mu_\ell \le \frac{C_{\rm opt}^{1+s}}{(1-q_{\rm lin}^{1/s})^s} \|\phi\|_{\mathbb{A}_s} (|\mathcal{K}_\ell| - |\mathcal{K}_0|)^{-s} \quad \textit{for all } \ell \in \mathbb{N}_0. \tag{3.7}$$

The constants $q_{\rm lin}, C_{\rm lin}$ depend only on $p, w_{\min}, w_{\max}, \gamma, \theta$, and $\check{\kappa}_{\max}$ from (M1). *The constant $\theta_{\rm opt}$ depends only on $p, w_{\min}, w_{\max}, \gamma$, and* (M1)–(M3), *and $C_{\rm opt}$ depends additionally on θ.*

Remark 3.3. *The proof of Theorem 3.2 reveals that linear convergence* (3.5) *only requires* (M1), *while optimal rates* (3.7) *rely on* (3.5) *and* (M2)–(M3).

The proof of our main result is split in the following two sections. The ideas essentially follow those of [CFPP14], where an axiomatic approach of adaptivity for abstract problems is found. We note, however, that [CFPP14] only considers h-refinement, while the present Algorithm 3.1 steers both, the h-refinement and the knot multiplicity increase.

Remark 3.4 (Convergence of Algorithm 3.1 for the Faermann estimator)**.** *In* [FGP15], *we analyzed the Faermann estimator $\eta_\star^2 := \sum_{z\in\mathcal{N}_\star} \eta_\star^2(z)$ with $\eta_\star^2(z) := |f - V\Phi_\star|^2_{H^{1/2}(\omega_\star(z))}$ which was first introduced for standard BEM in* [Fae00], *in the frame of adaptive IGA-BEM. Unlike the weighted-residual error estimator* (3.1), *the Faermann error estimator is well-defined under minimal regularity of the given data $f \in H^{1/2}(\Gamma)$. We proved that $\eta_\star$ is even efficient and reliable, i.e., $\eta_\star \simeq \|\phi - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)}$ and satisfies $\eta_\star(z) \lesssim \mu_\star(z)$; see* [FGP15, Theorem 3.1 and 4.4] *resp.* [FGHP15, Theorem 4.3]. *Instead of the weighted-residual estimator $\mu_\star$, Algorithm 3.1 can also be steered by the local contributions of $\eta_\star$. Numerical experiments in* [FGP15, FGHP15] *showed optimal convergence of this adaptive strategy. However, a mathematical proof for such an empirical observation is even open for h-adaptive standard BEM with piecewise polynomials. However,* [FFME+14, Theorem 3.2] *proves at least plain convergence for adaptive standard BEM.*

Indeed, adaptive IGABEM steered by the Faermann estimator fits exactly into the abstract framework of [FFME+14, Section 2]*: Instead of triangulations $\mathcal{T}_\ell$ and elements*

$T \in \mathcal{T}_\ell$, one can employ the argument for the sets of nodes $\mathcal{N}_\ell$ and the nodes $z \in \mathcal{N}_\ell$. As auxiliary estimator, we consider

$$\rho_\ell'^2 := \sum_{z \in \mathcal{N}_\ell} \rho_\ell'^2(z) \quad \textit{with} \quad \rho_\ell'^2(z) := \|\widetilde{h}_\ell^{1/2} \partial_\Gamma (f - V\Phi_\ell)\|^2_{L^2(\omega_\ell(z))},$$

where $\widetilde{h}_\ell$ is the equivalent mesh-size function of Proposition 4.2 below. The auxiliary estimator ρ'_ℓ and the Faermann estimator η_ℓ then satisfy the assumptions [FFME+14, (A1)–(A3)]*: Here, Assumption* (A1) *states that for any $f \in H^1(\Gamma)$, it holds*

$$\sum_{z \in \mathcal{M}_\ell} \eta_\ell^2(z) \le C_1 \sum_{z \in \mathcal{M}_\ell} \rho_\ell'^2(z) \quad \textit{for all } \ell \in \mathbb{N}_0$$

which is guaranteed by [FGHP15, Theorem 4.3]*. Assumption* (A2) *states that, for any $\delta > 0$, it holds*

$$C_2^{-1} \sum_{z \in \mathcal{M}_\ell} \rho_\ell'^2(z) \le \rho_\ell'^2 - \frac{1}{1+\delta} \rho_{\ell+m}'^2 + (1+\delta^{-1}) C_2 \|\Phi_{\ell+m} - \Phi_\ell\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \quad \textit{for all } \ell, m \in \mathbb{N}_0.$$

This can be proven essentially as in [FFME+14, Theorem 3.1]*. The heart of the matter is the estimate $\widetilde{h}_{\ell+1} \le q_{\rm ctr} \widetilde{h}_\ell$ on $\omega_\ell(\mathcal{M}_\ell)$, which follows from Proposition 4.2 below and*

$$\mathcal{M}_\ell \subset \bigcup([\mathcal{T}_\ell] \setminus [\mathcal{T}_{\ell+1}]).$$

Assumption (A3) *states that, for any $\widetilde{f} \in H^{1/2}(\Gamma)$ with corresponding solution $\widetilde{\phi} = V^{-1}\widetilde{f} \in \widetilde{H}^{-1/2}(\Gamma)$ and Galerkin approximation $\widetilde{\Phi}_\ell \in \mathcal{X}_\ell$, it holds*

$$\Big| \sum_{z \in \mathcal{M}_\ell} \eta_\ell^2(f, z) - \sum_{z \in \mathcal{M}_\ell} \eta_\ell^2(\widetilde{f}, z) \Big| \le C_3 \|f - \widetilde{f}\|_{H^{1/2}(\Gamma)} \quad \textit{for all } \ell \in \mathbb{N}_0,$$

where $\eta_\ell^2(f,z) := \eta_\ell^2(z)$ and $\eta_\star^2(\widetilde{f}, z) := |\widetilde{f} - V\widetilde{\Phi}_\ell|^2_{H^{1/2}(\omega_\ell(z))}$. Since η_ℓ is efficient and has semi-norm structure, this is verified as in [FFME+14, Section 2.4]*. The generic constants $C_1, C_2, C_3 > 0$ depend only on $\check{\kappa}_{\max}, p, w_{\min}, w_{\max}$, and γ. Altogether, the assumptions* [FFME+14, (A1)–(A2)] *are fulfilled on the dense subset $H^1(\Gamma)$ of $H^{1/2}(\Gamma)$. Therefore,* [FFME+14, Proposition 2.5] *applies and shows that Algorithm 3.1 driven by the Faermann estimator leads to plain convergence $\|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \simeq \eta_\ell \to 0$ for all $f \in H^{1/2}(\Gamma)$.* □

4. Proof of Theorem 3.2, linear convergence (3.5)

As an auxiliary result, we need the following inverse-type estimate for NURBS with respect to the fractional $\widetilde{H}^{-1/2}(\Gamma)$-norm. For piecewise polynomials, an analogous result is already found in [GHS05, Theorem 3.6] resp. [Geo08, Theorem 3.9]. Our proof is inspired by [DFG+04, Section 4.3], where a similar result is found for piecewise constant functions as well as for piecewise affine and globally continuous functions in 1D. For integer-order Sobolev norms, inverse estimates for NURBS are found in [BBdVC+06, Section 4].

Proposition 4.1. *Let $[\mathcal{T}_\star] \in [\mathbb{T}]$. Then, there is a constant $C_{\rm inv} > 0$ such that*

$$\|h_\star^{1/2} \partial_\Gamma (V\Psi_\star)\|_{L^2(\Gamma)} + \|h_\star^{1/2} \Psi_\star\|_{L^2(\Gamma)} \le C_{\rm inv} \|\Psi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} \quad \textit{for all } \Psi_\star \in \mathcal{X}_\star. \tag{4.1}$$

The constant $C_{\rm inv}$ only depends on $\check{\kappa}_{\max}, p, w_{\min}, w_{\max}$ and γ.

Proof. The proof is done in four steps. First, we show that $\|h_\star^{1/2}\psi\|_{L^2(\Gamma)} \lesssim \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}$ holds for all $\psi \in L^2(\Gamma)$ which satisfy a certain assumption. In the second step, we prove an auxiliary result for polynomials which is needed in the third one, where we show that all $\psi \in \mathcal{X}_\star$ satisfy the mentioned assumption. In last step, we apply a recent result of [AFF+15], which then concludes the proof.

Step 1: Let $\mathcal{X} \subset L^2(\Gamma)$ satisfy the following assumption: There exists a constant $q \in (0,1)$ such that for all $T \in \mathcal{T}_\star$ and all $\psi \in \mathcal{X}$ there exists some connected subset $\Delta(T,\psi) \subseteq T$ of length $|\Delta(T,\psi)| \geq q|T|$ such that ψ does not change its sign on $\Delta(T,\psi)$ and

$$\min_{x\in\Delta(T,\psi)} |\psi(x)| \geq q \max_{x\in T} |\psi(x)|. \tag{4.2}$$

Then, there exists a constant $C > 0$ which depends only on q and $\check{\kappa}_\star$, such that

$$\|h_\star^{1/2}\psi\|_{L^2(\Gamma)} \leq C\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} \quad \text{for all } \psi \in \mathcal{X}.$$

For a compact nonempty interval $[c,d] = I \subseteq [a,b]$, we define the bubble function

$$P_I(t) := \begin{cases} \left(\frac{t-c}{d-c} \cdot \frac{d-t}{d-c}\right)^2 & \text{if } t \in I, \\ 0 & \text{if } t \in [a,b] \setminus I. \end{cases}$$

It obviously satisfies $0 \leq P_I \leq 1$ and $\mathrm{supp} P_I = I$. A standard scaling argument proves

$$C_1|I| \leq \|P_I\|^2_{L^2(I)} \leq \|P_I\|_{L^1(I)} \leq C_2|I| \tag{4.3}$$

and

$$|I|^2\|P_I'\|^2_{L^2(I)} \leq C_3\|P_I\|^2_{L^2(I)} \tag{4.4}$$

with generic constants $C_1, C_2, C_3 > 0$ which do not depend on I. For each $T \in \mathcal{T}_\star$, let $I(T,\psi)$ be some interval with $\gamma(I(T,\psi)) = \Delta(T,\psi)$. With the arclength parametrization γ_L, we define, for all $T \in \mathcal{T}_\star$, the functions $P_{\Delta(T,\psi)} := P_{I(T,\psi)} \circ \gamma_L$ and the coefficients

$$c_T := \mathrm{sgn}(\psi|_{\Delta(T,\psi)}) h_{\star,T} \min_{x\in\Delta(T,\psi)} |\psi(x)|. \tag{4.5}$$

Note that (4.3)–(4.4) hold for $P_{\Delta(T,\psi)}$ with I simply replaced by $\Delta(T,\psi)$ and with $(\cdot)'$ replaced by the arclength derivative ∂_Γ. By definition of the dual norm it holds

$$\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} \geq \frac{|\langle \psi\,;\,\chi\rangle|}{\|\chi\|_{H^{1/2}(\Gamma)}} \quad \text{with, e.g., } \chi := \sum_{T\in\mathcal{T}_\star} c_T P_{\Delta(T,\psi)} \in H^1(\Gamma) \subset H^{1/2}(\Gamma). \tag{4.6}$$

First, we estimate the numerator in (4.6):

$$\begin{aligned} |\langle \psi\,;\,\chi\rangle| &= \left|\sum_{T\in\mathcal{T}_\star} \int_T \psi(x) c_T P_{\Delta(T,\psi)}(x)\,dx\right| \overset{(4.5)}{\geq} \sum_{T\in\mathcal{T}_\star} h_{\star,T} \min_{x\in\Delta(T,\psi)} |\psi(x)|^2\, \|P_{\Delta(T,\psi)}\|_{L^1(\Delta(T,\psi))} \\ &\overset{(4.2)}{\geq} q^2 \sum_{T\in\mathcal{T}_\star} h_{\star,T} \max_{x\in T} |\psi(x)|^2 \|P_{\Delta(T,\psi)}\|_{L^1(\Delta(T,\psi))} \\ &\overset{(4.3)}{\geq} C_1 q^3 \sum_{T\in\mathcal{T}_\star} h_{\star,T}\, \|\psi\|^2_{L^2(T)} \\ &= C_1 q^3 \|h_\star^{1/2}\psi\|^2_{L^2(\Gamma)}. \end{aligned}$$

It remains to estimate the denominator in (4.6): With [Fae00, Lemma 2.3] and [FGHP15, Lemma 4.5], we see

$$
\begin{aligned}
|\chi|^2_{H^{1/2}(\Gamma)} &\overset{\text{[Fae00]}}{\lesssim} \|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)} + \sum_{z\in\mathcal{N}_\star} |\chi|^2_{H^{1/2}(\omega_z)} \\
&\overset{\text{[FGHP15]}}{\lesssim} \|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)} + \sum_{z\in\mathcal{N}_\star} \|h_\star^{1/2}\partial_\Gamma\chi\|^2_{L^2(\omega_z)} \\
&\simeq \|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)} + \sum_{T\in\mathcal{T}_\star} \|h_\star^{1/2}\partial_\Gamma\chi\|^2_{L^2(\Delta(T,\psi))} \\
&= \|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)} + \sum_{T\in\mathcal{T}_\star} h_{\star,T} c_T^2 \|\partial_\Gamma P_{\Delta(T,\psi)}\|^2_{L^2(\Delta(T,\psi))} \\
&\overset{(4.4)}{\le} \|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)} + C_3 \sum_{T\in\mathcal{T}_\star} h_{\star,T} c_T^2 |\Delta(T,\psi)|^{-2} \|P_{\Delta(T,\psi)}\|^2_{L^2(\Delta(T,\psi))} \\
&\simeq \|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)}.
\end{aligned}
$$

This yields

$$
\|\chi\|^2_{H^{1/2}(\Gamma)} = \|\chi\|^2_{L^2(\Gamma)} + |\chi|^2_{H^{1/2}(\Gamma)} \lesssim \|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)},
$$

where the hidden constant depends only on $\check{\kappa}_{\max}$ and γ. With

$$
\begin{aligned}
\|h_\star^{-1/2}\chi\|^2_{L^2(\Gamma)} &= \sum_{T\in\mathcal{T}_\star} h_{\star,T}^{-1} c_T^2 \|P_{\Delta(T,\psi)}\|^2_{L^2(\Delta(T,\psi))} \overset{(4.3)}{\le} C_2 \sum_{T\in\mathcal{T}_\star} h_{\star,T}^{-1} c_T^2 |\Delta(T,\psi)| \\
&\overset{(4.5)}{=} C_2 \sum_{T\in\mathcal{T}_\star} h_{\star,T} \min_{x\in\Delta(T,\psi)} |\psi(x)|^2 |\Delta(T,\psi)| \\
&\le C_2 \sum_{T\in\mathcal{T}_\star} h_{\star,T} \|\psi\|^2_{L^2(\Delta(T,\psi))} \le C_2 \|h_\star^{1/2}\psi\|^2_{L^2(\Gamma)}
\end{aligned}
$$

we finish the first step.

Step 2: For some fixed polynomial degree $p \in \mathbb{N}_0$, there exists a constant $q_1 \in (0,1)$ such that for all polynomials F of degree p on $[0,1]$ there exists some interval $I \subseteq [0,1]$ of length $|I| \ge q_1$ with

$$
\min_{t\in I} |F(t)| \ge q_1 \max_{t\in[0,1]} |F(t)|. \tag{4.7}
$$

Instead of considering general polynomials $\mathcal{P}^p([0,1])$ of degree p, it is sufficient to consider the following subset

$$
\mathcal{M} := \big\{ F \in \mathcal{P}^p([0,1]) \,:\, \|F\|_\infty = 1 \big\}.
$$

Note that $\mathcal{M}$ is a compact subset of $L^\infty([0,1])$ and that differentiation $(\cdot)'$ is a continuous mapping on $\mathcal{M}$ due to finite dimension. This especially implies boundedness $\sup_{F\in\mathcal{M}} \|F'\|_\infty \le C_4 < \infty$. We may assume $C_4 > 2$. For given $F \in \mathcal{M}$, we define the interval I as follows: Without loss of generality, we assume that the maximum of $|F|$ is attained at some $t_1 \in [0,1/2]$ and that $F(t_1) = 1$. We set $t_3 := t_1 + C_4^{-1} \in (t_1, 1]$ and $t_2 := t_1 + C_4^{-1}/2 \in (t_1, 3/4]$ and $I := [t_1, t_2]$. Then, $|I| = 1/(2C_4)$ and for all $t \in I$ it holds

$$
1/2 \le C_4(t_3 - t) = F(t_1) + C_4(t_1 - t) \le F(t_1) + \|F'\|_\infty (t_1 - t) \le F(t) = |F(t)|.
$$

Altogether, we thus have

$$q_1 := 1/(2C_4) \le 1/2 \le \min_{t\in I} |F(t)| \quad \text{and} \quad |I| = q_1$$

and conclude this step.
Step 3: We show that $\mathcal{X}_\star$ satisfies the assumption of Step 1 and hence conclude $\|h_\star^{1/2}\Psi_\star\|_{L^2(\Gamma)} \lesssim \|\Psi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)}$ for all $\Psi_\star \in \mathcal{X}_\star$: Let $\check{T} \subset [a,b]$ be the interval with $\gamma(\check{T}) = T$ and $\check{\psi} := \psi \circ \gamma|_{\check{T}}$. Since $|I| \simeq |\gamma(I)|$ for any interval $I \subseteq [a,b]$, where the hidden constants depend only on γ, we just have to find a uniform constant $q_2 \in (0,1)$ and some interval $I \subseteq \check{T}$ of length $|I| \ge q_2|\check{T}|$ with

$$\min_{t\in I} |\check{\psi}(t)| \ge q_2 \max_{x\in\check{T}} |\check{\psi}(t)|. \tag{4.8}$$

The function $\check{\psi}$ has the form F/w with a polynomial F of degree p and the weight function w, which is also a polynomial of degree p and which satisfies $w_{\min} \le w \le w_{\max}$. Hence, (4.8) is especially satisfied if

$$\min_{t\in I} |F(t)| \ge q_1 \frac{w_{\max}}{w_{\min}} \max_{x\in\check{T}} |F(t)|. \tag{4.9}$$

After scaling to the interval $[0,1]$, we can apply Step 2 and conclude this step.
Step 4: According to [AFF$^+$15], it holds $\|h_\star^{1/2}\partial_\Gamma(V\psi)\|_{L^2(\Gamma)} \lesssim \|h_\star^{1/2}\psi\|_{L^2(\Gamma)} + \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}$ for all $\psi \in L^2(\Gamma)$, where the hidden constant depends only on Γ, γ, and $\check{\kappa}_{\max}$. Together with Step 3, this shows (4.1). □

The proof of linear convergence (3.5) will be done with the help of some auxiliary (and purely theoretical) error estimator $\widetilde{\rho}_\star$. The latter relies on the following definition of an equivalent mesh-size function which respects the multiplicity of the knots.

Proposition 4.2. *Assumption 2.1* (M1) *implies the existence of a modified mesh-size function $\widetilde{h} : [\mathbb{T}] \to L^\infty(\Gamma)$ with the following properties: There exists a constant $C_{\mathrm{wt}} > 0$ and $0 < q_{\mathrm{ctr}} < 1$ which depend only on $\check{\kappa}_{\max}, p$ and γ such that for all $[\mathcal{T}_\star] \in [\mathbb{T}]$ and all refinements $[\mathcal{T}_+] \in$ `ref`$([\mathcal{T}_\star])$, the corresponding mesh-sizes $\widetilde{h}_\star := \widetilde{h}([\mathcal{T}_\star])$ and $\widetilde{h}_+ := \widetilde{h}([\mathcal{T}_+])$ satisfy equivalence*

$$C_{\mathrm{wt}}^{-1}\check{h}_\star \le \widetilde{h}_\star \le C_{\mathrm{wt}}\check{h}_\star, \tag{4.10}$$

reduction

$$\widetilde{h}_+ \le \widetilde{h}_\star, \tag{4.11}$$

as well as contraction on the patch of refined elements

$$\widetilde{h}_+|_{\omega_+([\mathcal{T}_+]\setminus[\mathcal{T}_\star])} \le q_{\mathrm{ctr}}\widetilde{h}_\star|_{\omega_+([\mathcal{T}_+]\setminus[\mathcal{T}_\star])}. \tag{4.12}$$

Note that $\omega_+([\mathcal{T}_+]\setminus[\mathcal{T}_\star]) = \omega_\star([\mathcal{T}_\star]\setminus[\mathcal{T}_+])$, which follows from $\bigcup([\mathcal{T}_+]\setminus[\mathcal{T}_\star]) = \bigcup([\mathcal{T}_\star]\setminus[\mathcal{T}_+])$ and the fact that the application of ω_+ resp. $\omega_\star$ only adds elements of $\mathcal{T}_\star \cap \mathcal{T}_+$.

Proof. For all $[\mathcal{T}_\star] \in \mathbb{T}$, we define $\widetilde{h}_\star \in L^\infty(\Gamma)$ by

$$\widetilde{h}_\star|_T = |\gamma^{-1}(\omega_\star(T))| \cdot q_1^{\sum_{z\in\mathcal{N}_\star\cap\omega_\star(T)} \#z} \quad \text{for all } T \in \mathcal{T}_\star,$$

where $0 < q_1 < 1$ is fixed later. Clearly, $\widetilde{h}_\star \simeq \check{h}_\star$, where the hidden equivalence constants depend only on $\check{\kappa}_\star$, p, and q_1. Let $x \in \Gamma$. First, suppose $x \notin \omega_+([\mathcal{T}_+] \setminus [\mathcal{T}_\star]) \cup \mathcal{N}_+$, i.e., neither the element $[T] \in [\mathcal{T}_+]$ containing x nor its neighbors result from h-refinement or from multiplicity increase. Then, $\widetilde{h}_+(x) = \widetilde{h}_\star(x)$. Second, suppose $x \in \omega_+([\mathcal{T}_+]\setminus[\mathcal{T}_\star])\setminus\mathcal{N}_+$, i.e., the element $[T'] \in [\mathcal{T}_+]$ containing x or one of its neighbors result from h-refinement and/or multiplicity increase. If only multiplicity increase took place, we get

$$q_1^{\sum_{z\in\mathcal{N}_+\cap\omega_+(T')}\#z} \le q_1 \cdot q_1^{\sum_{z\in\mathcal{N}_\star\cap\omega_\star(T)}\#z}.$$

In the other case, consider the father $[T] \in [\mathcal{T}_\star]$ of $[T']$, i.e., $T' \subseteq T$. Note that

$$|\gamma^{-1}(\omega_+(T'))| \le q_2\,|\gamma^{-1}(\omega_\star(T))|$$

with a constant $0 < q_2 < 1$ which depends only on $\check{\kappa}_{\max}$. Choose $0 < q_1 < 1$ sufficiently large such that

$$q_2/q_1^{4p} < 1.$$

This choice yields $\widetilde{h}_+(x) \le (q_2/q_1^{4p}) \cdot \widetilde{h}_\star(x)$, since $\mathcal{N}_\star \cap \omega_\star(T)$ contains at most 4 nodes. Therefore, we conclude the proof with $q_{\rm ctr} := \max(q_1, q_2/q_1^{4p})$. □

Remark 4.3. *Note that the construction of $\widetilde{h}_\star$ in Proposition 4.2 even ensures contraction $\widetilde{h}_+|_{\omega_+(T)} \le q_{\rm ctr}\widetilde{h}_\star|_{\omega_+(T)}$ if $[T] \in [\mathcal{T}_+] \setminus [\mathcal{T}_\star]$ is obtained by h-refinement, while the multiplicity of all nodes $z \in \mathcal{N}_+ \cap \omega_+(T)$ is arbitrarily chosen $\#z \in \{1, \dots, p+1\}$. In explicit terms, this allows for instance to set the multiplicity of all nodes $z \in \mathcal{N}_+ \cap \omega_+(T)$ to $\#z := 1$, if T is obtained by h-refinement.* □

For any $[\mathcal{T}_\star] \in [\mathbb{T}]$, we define the auxiliary estimator

$$\widetilde{\rho}_\star^2 := \sum_{T\in\mathcal{T}} \widetilde{\rho}_\star^2(T) \quad \text{with} \quad \widetilde{\rho}_\star^2(T) := \|\widetilde{h}_\star^{1/2}\partial_\Gamma(f - V\Phi_\star)\|_{L^2(T)}^2 \tag{4.13}$$

which employs the novel mesh-size function $\widetilde{h}_\star$ from Proposition 4.2. Obviously the estimators $\mu_\star$ and $\widetilde{\rho}_\star$ are locally equivalent

$$\widetilde{\rho}_\star^2(T) \lesssim \mu_\star^2(z) \lesssim \sum_{\substack{T'\in\mathcal{T}_\star \\ z\in T'}} \widetilde{\rho}_\star^2(T') \quad \text{for all } z \in \mathcal{N}_\star \text{ and } T \in \mathcal{T}_\star \text{ with } z \in T, \tag{4.14}$$

where the hidden constants depend only on $\check{\kappa}_{\max}$, p, and γ. The proof of the following lemma is inspired by [FKMP13, Proposition 3.2] resp. [CFPP14, Lemma 8.8], where only h-refinement is considered.

Lemma 4.4 (estimator reduction of $\widetilde{\rho}$)**.** *Algorithm 3.1 guarantees*

$$\widetilde{\rho}_{\ell+1}^2 \le q_{\rm est}\widetilde{\rho}_\ell^2 + C_{\rm est}\|\Phi_{\ell+1} - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 \quad \textit{for all } \ell \ge 0. \tag{4.15}$$

The constants $0 < q_{\rm est} < 1$ and $C_{\rm est} > 0$ depend only on $\check{\kappa}_{\max}, p, w_{\min}, w_{\max}, \gamma$, and θ.

Proof. The proof is done in several steps.
Step 1: With the inverse estimate (4.1), there holds the following stability property for any measurable $\Gamma_0 \subseteq \Gamma$

$$\begin{aligned}\Big|\|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma(f - V\Phi_{\ell+1})\|_{L^2(\Gamma_0)} - \|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma(f - V\Phi_\ell)\|_{L^2(\Gamma_0)}\Big| \\ \le \|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma V(\Phi_{\ell+1} - \Phi_\ell)\|_{L^2(\Gamma_0)} \le C\|\Phi_{\ell+1} - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)},\end{aligned}$$

with a constant $C > 0$ which depends only on $C_{\rm wt}, C_{\rm inv}$, and γ.
Step 2: With Proposition 4.2, we split the estimator into a contracting and into a non-contracting part

$$\widetilde{\rho}_{\ell+1}^2 = \|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma(f - V\Phi_{\ell+1})\|_{L^2(\omega_{\ell+1}([\mathcal{T}_{\ell+1}]\setminus[\mathcal{T}_\ell]))}^2 + \|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma(f - V\Phi_{\ell+1})\|_{L^2(\Gamma\setminus\omega_{\ell+1}([\mathcal{T}_{\ell+1}]\setminus[\mathcal{T}_\ell]))}$$

Step 1, the Young inequality, and Proposition 4.2 show, for arbitrary $\delta > 0$, that

$$\begin{aligned}
&\|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma(f - V\Phi_{\ell+1})\|^2_{L^2(\omega_{\ell+1}([\mathcal{T}_{\ell+1}]\setminus[\mathcal{T}_\ell]))} \\
&\quad \le (1+\delta)\|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma(f - V\Phi_\ell)\|^2_{L^2(\omega_{\ell+1}([\mathcal{T}_{\ell+1}]\setminus[\mathcal{T}_\ell]))} + (1+\delta^{-1})C^2\|\Phi_{\ell+1} - \Phi_\ell\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \\
&\quad \le (1+\delta)q_{\rm ctr}\|\widetilde{h}_{\ell}^{1/2}\partial_\Gamma(f - V\Phi_\ell)\|^2_{L^2(\omega_{\ell}([\mathcal{T}_{\ell}]\setminus[\mathcal{T}_{\ell+1}]))} + (1+\delta^{-1})C^2\|\Phi_{\ell+1} - \Phi_\ell\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.
\end{aligned}$$

Analogously, we get

$$\begin{aligned}
&\|\widetilde{h}_{\ell+1}^{1/2}\partial_\Gamma(f - V\Phi_{\ell+1})\|^2_{L^2(\Gamma\setminus\omega_{\ell+1}([\mathcal{T}_{\ell+1}]\setminus[\mathcal{T}_\ell]))} \\
&\quad \le (1+\delta)\|\widetilde{h}_{\ell}^{1/2}\partial_\Gamma(f - V\Phi_\ell)\|^2_{L^2(\Gamma\setminus\omega_{\ell}([\mathcal{T}_{\ell}]\setminus[\mathcal{T}_{\ell+1}]))} + (1+\delta^{-1})C^2\|\Phi_{\ell+1} - \Phi_\ell\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.
\end{aligned}$$

Combining these estimates, we end up with

$$\begin{aligned}
\widetilde{\rho}_{\ell+1}^2 \le (1+\delta)\widetilde{\rho}_\ell^2 - (1+\delta)(1-q_{\rm ctr})\|\widetilde{h}_{\ell}^{1/2}\partial_\Gamma(f - V\Phi_\ell)\|^2_{L^2(\omega_{\ell}([\mathcal{T}_{\ell}]\setminus[\mathcal{T}_{\ell+1}]))} \\
+ 2(1+\delta^{-1})C^2\|\Phi_{\ell+1} - \Phi_\ell\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.
\end{aligned} \tag{4.16}$$

Step 3: Local equivalence (4.14) and the Dörfler marking (3.3) for μ_ℓ imply

$$\theta\widetilde{\rho}_\ell^2 \simeq \theta\mu_\ell^2 \le \sum_{z\in\mathcal{M}_\ell}\mu_\ell(z)^2 \simeq \sum_{\substack{T\in\mathcal{T}_\ell \\ T\subseteq\omega_\ell(\mathcal{M}_\ell)}}\widetilde{\rho}_\ell(T)^2,$$

where the hidden constants depend only on $\check{\kappa}_{\max}$, p, and γ. Hence, $\widetilde{\rho}_\ell$ satisfies some Dörfler marking with a certain parameter $0 < \widetilde{\theta} < 1$. With $\mathcal{M}_\ell \subseteq \bigcup([\mathcal{T}_\ell] \setminus [\mathcal{T}_{\ell+1}])$, (4.16) hence becomes

$$\widetilde{\rho}_{\ell+1}^2 \le \Big((1+\delta) - (1+\delta)(1-q_{\rm ctr})\widetilde{\theta}\Big)\,\widetilde{\rho}_\ell^2 + 2(1+\delta^{-1})C^2\|\Phi_{\ell+1} - \Phi_\ell\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Choosing δ sufficiently small, we prove (4.15) with $C_{\rm est} := 2(1+\delta^{-1})C^2$ and $q_{\rm est} := (1+\delta)\big(1-(1-q_{\rm ctr})\widetilde{\theta}\big) < 1$. $\square$

Proof of linear convergence (3.5). Due to the properties of the weakly-singular integral operator V, the bilinear form $A(\phi,\psi) := \langle V\phi\,;\,\psi\rangle_\Gamma$ defines even a scalar product, and the induced norm $\|\psi\|_V := A(\psi,\psi)^{1/2}$ is an equivalent norm on $\widetilde{H}^{-1/2}(\Gamma)$. According to nestedness of the ansatz spaces $\mathcal{X}_\ell \subset \mathcal{X}_{\ell+1}$, the Galerkin orthogonality implies the Pythagoras theorem

$$\|\phi - \Phi_{\ell+1}\|_V^2 + \|\Phi_{\ell+1} - \Phi_\ell\|_V^2 = \|\phi - \Phi_\ell\|_V^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Together with the estimator reduction (4.15) and reliability (3.2)

$$\|\phi - \Phi_\ell\|_V \simeq \|\phi - \Phi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \mu_\ell \simeq \widetilde{\rho}_\ell,$$

this implies the existence of $0 < \kappa, \lambda < 1$, which depend only on $C_{\rm rel}, C_{\rm est}$ and $q_{\rm est}$, such that $\Delta_\star := \|\phi - \Phi_\star\|_V^2 + \lambda\,\widetilde{\rho}_\star^2 \simeq \widetilde{\rho}_\star^2$ satisfies

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \text{for all } \ell \in \mathbb{N}_0;$$

see, e.g., [FKMP13, Theorem 4.1], while the original idea goes back to [CKNS08]. From this, we infer

$$\mu_{\ell+n}^2 \simeq \widetilde{\rho}_{\ell+n}^2 \simeq \Delta_{\ell+n} \le \kappa^n\,\Delta_\ell \simeq \kappa^n\,\widetilde{\rho}_\ell^2 \simeq \kappa^n\,\mu_\ell^2 \quad \text{for all } \ell, n \in \mathbb{N}_0$$

and hence conclude the proof. $\square$

5. Proof of Theorem 3.2, optimal convergence (3.7)

As in the previous section, we define an auxiliary error estimator. For each $[\mathcal{T}_\star] \in [\mathbb{T}]$, let

$$\rho_\star^2 := \sum_{T\in\mathcal{T}} \rho_\star(T)^2 \quad \text{with} \quad \rho_\star(T)^2 := \|\check{h}_\star^{1/2}\partial_\Gamma(f - V\Phi_\star)\|_{L^2(T)}^2. \tag{5.1}$$

Note that the estimators $\mu_\star$ and $\rho_\star$ are again locally equivalent

$$\rho_\star^2(T) \le \mu_\star^2(z) \lesssim \sum_{\substack{T'\in\mathcal{T}_\star\\ z\in T'}} \rho_\star^2(T') \quad \text{for all } z \in \mathcal{N}_\star \text{ and } T \in \mathcal{T}_\star \text{ with } z \in T, \tag{5.2}$$

where the hidden constant depends only on $\check{\kappa}_{\max}$. Analogous versions of the next two lemmas are already proved in [FKMP13, Proposition 4.2 and 4.3] for h-refinement and piecewise constants; see also [CFPP14, Propostion 5.7] for discontinuous piecewise polynomials and h-refinement. The proof for Lemma 5.1 is essentially based on Proposition 4.1. The proof of Lemma 5.2 requires the construction of a Scott-Zhang type operator (5.9) which is not necessary in [FKMP13, CFPP14], since both works consider discontinuous piecewise polynomials.

Lemma 5.1 (stability of ρ)**.** *Let $[\mathcal{T}_\star] \in [\mathbb{T}]$ and $[\mathcal{T}_+] \in \mathtt{ref}(\mathcal{T}_\star)$. For $\mathcal{S} \subseteq \mathcal{T}_\star \cap \mathcal{T}_+$ there holds*

$$\Big|\Big(\sum_{T\in\mathcal{S}} \rho_+(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{S}} \rho_\star(T)^2\Big)^{1/2}\Big| \le C_{\mathrm{stab}}\|\Phi_+ - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)}, \tag{5.3}$$

where $C_{\mathrm{stab}} > 0$ depends only on the parametrization γ and the constant C_{inv} of Proposition 4.1

Proof. For all subsets $\Gamma_0 \subseteq \Gamma$, it holds

$$\begin{aligned}\Big|\|\check{h}_+^{1/2}\partial_\Gamma(f - V\Phi_+)\|_{L^2(\Gamma_0)} - \|\check{h}_+^{1/2}\partial_\Gamma(f - V\Phi_\star)\|_{L^2(\Gamma_0)}\Big| &\le \|\check{h}_+^{1/2}\partial_\Gamma V(\Phi_+ - \Phi_\star)\|_{L^2(\Gamma_0)}\\ \lesssim \|h_+^{1/2}\partial_\Gamma V(\Phi_+ - \Phi_\star)\|_{L^2(\Gamma_0)} \le C_{\mathrm{inv}}\|\Phi_+ - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)}.&\end{aligned} \tag{5.4}$$

The choice $\Gamma_0 = \bigcup\mathcal{S}$ shows stability

$$\Big|\Big(\sum_{T\in\mathcal{S}} \rho_+(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{S}} \rho_\star(T)^2\Big)^{1/2}\Big| \le C_{\mathrm{inv}}\|\Phi_+ - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)},$$

and we conclude the proof. □

Lemma 5.2 (discrete reliability of ρ)**.** *There exist constants $C_{\mathrm{rel}}, C_{\mathrm{ref}} > 0$, which depend only on $\check{\kappa}_{\max}, p, w_{\min}, w_{\max}$, and γ, such that for all refinements $[\mathcal{T}_+] \in \mathtt{ref}([\mathcal{T}_\star])$ of $[\mathcal{T}_\star] \in [\mathbb{T}]$ there exists a subset $\mathcal{R}_\star(\mathcal{T}_+) \subseteq \mathcal{T}_\star$ with*

$$\|\Phi_+ - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 \le C_{\mathrm{rel}} \sum_{T\in\mathcal{R}_\star(\mathcal{T}_+)} \rho_\star(T)^2 \tag{5.5}$$

as well as

$$\bigcup([\mathcal{T}_\star] \setminus [\mathcal{T}_+]) \subseteq \bigcup\mathcal{R}_\star(\mathcal{T}_+) \quad \textit{and} \quad |\mathcal{R}_\star(\mathcal{T}_+)| \le C_{\mathrm{ref}}|[\mathcal{T}_\star] \setminus [\mathcal{T}_+]|. \tag{5.6}$$

For the proof of Lemma 5.2, we need to introduce a Scott-Zhang type operator. Let $[\mathcal{T}_\star] \in [\mathbb{T}]$ and $\big\{R_{i,p}|_{[a,b\rangle} : i = 1-p, \dots, N-p\big\} \circ \gamma|_{[a,b\rangle}^{-1}$ be the basis of NURBS of $\mathcal{X}_\star$, where "$\rangle$" stands for "$)$" if $\Gamma = \partial\Omega$ is closed and for "$]$" if $\Gamma \subsetneqq \partial\Omega$ is open. Here, N denotes the number of transformed knots $\check{\mathcal{K}}_\star$ in $(a, b]$. With the corresponding B-splines there holds $R_{i,p} = B_{i,p}/w$, where $w = \sum_{\ell\in\mathbb{Z}} w_\ell B_{\ell,p}$ is the fixed denominator satisfying

$w_{\min} \le w \le w_{\max}$; see Section 2.8. In [BdVBSV14, Section 2.1.5], it is shown that, for $i \in \{1-p, \dots, N-p\}$, there exist dual basis functions $B^*_{i,p} \in L^2(\mathrm{supp} B_{i,p})$ with

$$\int_{\mathrm{supp} B_{i,p}} B^*_{i,p}(t) B_{j,p}(t)dt = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} \tag{5.7}$$

and

$$\|B^*_{i,p}\|_{L^2(\mathrm{supp} B_{i,p})} \le (2p+3)9^p |\mathrm{supp} B_{i,p}|^{-1/2}. \tag{5.8}$$

Define $R^*_{i,p} := B^*_{i,p} w$ with the denominator w from before, and $\widehat{R}_{i,p} := R_{i,p}|_{[a,b\rangle} \circ \gamma|^{-1}_{[a,b\rangle}$. For $I \subseteq \{1-p, \dots, N-p\}$, we define the following Scott-Zhang type operator

$$P_{\star,I} : L^2(\Gamma) \to \mathcal{X}_\star : \psi \mapsto \sum_{i \in I} \Big(\int_{\mathrm{supp} R_{i,p}} R^*_{i,p}(t) \psi(\gamma(t))\, dt \Big) \widehat{R}_{i,p}. \tag{5.9}$$

In [BdVBSV14, Section 3.1.2], a similar operator is considered for $I = \{1-p, \dots, N-p\}$, and [BdVBSV14, Proposition 2.2] proves an analogous version of the following lemma.

Lemma 5.3. *The Scott-Zhang type operator* (5.9) *satisfies the following two properties:*

(i) *Local projection property: For $T \in \mathcal{T}_\star$ with $\big\{i : T \subseteq \mathrm{supp} \widehat{R}_{i,p}\big\} \subseteq I$ and $\psi \in L^2(\Gamma)$, the inclusion $\psi|_{\omega^p_\star(T)} \in \mathcal{X}_\star|_{\omega^p_\star(T)} := \big\{\xi|_{\omega^p_\star(T)} : \xi \in \mathcal{X}_\star\big\}$ implies $\psi|_T = (P_{\star,I}\psi)|_T$.*

(ii) *Local L^2-stability: For $\psi \in L^2(\Gamma)$ and $T \in \mathcal{T}_\star$, there holds*

$$\|P_{\star,I}(\psi)\|_{L^2(T)} \le C_{\mathrm{sz}} \|\psi\|_{L^2(\omega^p_\star(T))},$$

where C_{sz} depends only on $\check{\kappa}_{\max}, p, w_{\max}$, and γ.

Proof. All NURBS basis functions which are non-zero on T, have support in $\omega^p_\star(T)$. With this, (i) follows easily from the definition of $P_{\star,I}$. For stability (ii), we use $0 \le \widehat{R}_{i,p} \le 1$ and (5.8) to see

$$\begin{aligned}
\|P_{\star,I}\psi\|_{L^2(T)} &= \Big\| \sum_{i \in I} \Big(\int_{\mathrm{supp} R_{i,p}} R^*_{i,p}(t)\psi(\gamma(t))\, dt \Big) \widehat{R}_{i,p} \Big\|_{L^2(T)} \\
&\le \sum_{\substack{i \in I \\ |\mathrm{supp} \widehat{R}_{i,p} \cap T| > 0}} \Big| \int_{\mathrm{supp} R_{i,p}} R^*_{i,p}(t)\psi(\gamma(t))\, dt \Big| \|\widehat{R}_{i,p}\|_{L^2(T)} \\
&\lesssim \sum_{\substack{i \in I \\ |\mathrm{supp} \widehat{R}_{i,p} \cap T| > 0}} \|R^*_{i,p}\|_{L^2(\mathrm{supp} R_{i,p})} \|\psi\|_{L^2(\mathrm{supp} \widehat{R}_{i,p})} h^{1/2}_{\star,T} \\
&\overset{(5.8)}{\lesssim} \sum_{\substack{i \in I \\ |\mathrm{supp} \widehat{R}_{i,p} \cap T| > 0}} \|\psi\|_{L^2(\mathrm{supp} \widehat{R}_{i,p})} \lesssim \|\psi\|_{L^2(\omega^p_\star(T))}.
\end{aligned}$$

Overall, the hidden constants depend only on $\check{\kappa}_{\max}, p, w_{\max}$, and γ. □

Proof of Lemma 5.2. We choose

$$I := \big\{ i : |\mathrm{supp} \widehat{R}_{i,p} \cap \Gamma \setminus \omega^p_\star([\mathcal{T}_\star] \setminus [\mathcal{T}_+])| > 0 \big\}.$$

We prove that

$$P_{\star,I}(\Phi_+ - \Phi_\star) = \begin{cases} \Phi_+ - \Phi_\star & \text{on } \Gamma \setminus \omega^p_\star([\mathcal{T}_\star] \setminus [\mathcal{T}_+]), \\ 0 & \text{on } \bigcup([\mathcal{T}_\star] \setminus [\mathcal{T}_+]). \end{cases} \tag{5.10}$$

To see this, let $T \in \mathcal{T}_\star$ with $T \subseteq \Gamma \backslash \omega_\star^p([\mathcal{T}_\star] \backslash [\mathcal{T}_+])$. Then ,$\{i \,:\, T \subseteq \mathrm{supp} \widehat{R}_{i,p}\} \subseteq I$. It holds $\omega_\star^p(T) \subseteq \bigcup([\mathcal{T}_\star] \cap [\mathcal{T}_+])$, wherefore $(\Phi_+ - \Phi_\star)|_{\omega_\star^p(T)} \in \mathcal{X}_\star|_{\omega_\star^p(T)}$. Hence Lemma 5.3 (i) is applicable and proves $P_{\star,I}(\Phi_+ - \Phi_\star)|_T = (\Phi_\star - \Phi_\star)|_T$. For $T \in \mathcal{T}_\star$ with $T \subseteq \bigcup([\mathcal{T}_\star] \backslash [\mathcal{T}_+])$, the assertion follows immediately from the definition of $P_{\star,I}$, since $\hat{R}_{i,p}|_T = 0$ for $i \in I$.

Let $\widetilde{\mathcal{N}}_\star := \{z \in \mathcal{N}_\star \,:\, z \in \omega_\star^p([\mathcal{T}_\star] \setminus [\mathcal{T}_+])\}$. For $z \in \widetilde{\mathcal{N}}_\star$, let φ_z be the P^1 hat function, i.e., $\phi_z(z') = \delta_{zz'}$ for all $z' \in \mathcal{N}_\star$, $\mathrm{supp}(\phi_z) = \omega_\star(z)$, and $\partial_\Gamma \varphi_z = \mathrm{const.}$ on $T_{z,1}$ and $T_{z,2}$, where $\omega_\star(z) = T_{z,1} \cup T_{z,2}$ with $T_{z,1}, T_{z,2} \in \mathcal{T}_\star$. Because of Galerkin orthogonality and $\sum_{z \in \widetilde{\mathcal{N}}_\star} \varphi_z = 1$ on $\omega_\star^p([\mathcal{T}_\star] \setminus [\mathcal{T}_+])$, we see

$$\begin{aligned}\|\Phi_+ - \Phi_\star\|_V^2 &= \langle f - V\Phi_\star \,;\, (1 - P_{\star,I})(\Phi_+ - \Phi_\star)\rangle_\Gamma \\ &= \Big\langle \sum_{z \in \widetilde{\mathcal{N}}_\star} \varphi_z (f - V\Phi_\star); (1 - P_{\star,I})(\Phi_+ - \Phi_\star)\Big\rangle_\Gamma.\end{aligned}$$

We abbreviate $\Sigma := \sum_{z \in \widetilde{\mathcal{N}}} \varphi_z(f - V\Phi_\star)$ and estimate with (M1), Lemma 5.3 (ii) and Proposition 4.1

$$\begin{aligned}\langle \Sigma \,;\, P_{\star,I}(\Phi_+ - \Phi_\star)\rangle \quad &\le \quad \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)} \|h_\star^{1/2} P_{\star,I}(\Phi_+ - \Phi_\star)\|_{L^2(\Gamma)} \\ &\overset{(5.10)}{=} \quad \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)} \|h_+^{1/2} P_{\star,I}(\Phi_+ - \Phi_\star)\|_{L^2(\bigcup([\mathcal{T}_\star] \cap [\mathcal{T}_+]))} \\ &\overset{\text{Lem. } 5.3}{\lesssim} \quad \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)} \|h_+^{1/2}(\Phi_+ - \Phi_\star)\|_{L^2(\omega_\star^p([\mathcal{T}_\star] \cap [\mathcal{T}_+]))} \\ &\overset{\text{Prop. } 4.1}{\lesssim} \quad \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)} \|\Phi_+ - \Phi_\star\|_V,\end{aligned}$$

as well as

$$\langle \Sigma \,;\, \Phi_+ - \Phi_\star\rangle \le \|\Sigma\|_{H^{1/2}(\Gamma)} \|\Phi_+ - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} \simeq \|\Sigma\|_{H^{1/2}(\Gamma)} \|\Phi_+ - \Phi_\star\|_V.$$

So far, we thus have proved

$$\begin{aligned}\|\Phi_+ - \Phi_\star\|_V &\le \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)} + \|\Sigma\|_{H^{1/2}(\Gamma)} \\ &\le \|h_\star^{-1/2}(f - V\Phi_\star)\|_{L^2(\omega^{p+1}([\mathcal{T}_\star] \backslash [\mathcal{T}_+]))} + \|\Sigma\|_{H^{1/2}(\Gamma)}.\end{aligned} \tag{5.11}$$

Next, we use [FGP15, Lemma 3.4], [FGHP15, Lemma 4.5], and $|\partial_\Gamma \varphi_z| \simeq |\omega_\star(z)|^{-1}$ to estimate

$$\begin{aligned}\|\Sigma\|_{H^{1/2}(\Gamma)}^2 \quad &\overset{[\text{FGP15}]}{\lesssim} \quad \sum_{z \in \mathcal{N}_\star} |\Sigma|_{H^{1/2}(\omega_\star(z))}^2 + \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)}^2 \\ &\overset{[\text{FGHP15}]}{\lesssim} \quad \sum_{z \in \mathcal{N}_\star} \|h_\star^{1/2}\partial_\Gamma \Sigma\|_{L^2(\omega_\star(z))}^2 + \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)}^2 \\ &\lesssim \quad \|h_\star^{1/2}\partial_\Gamma\Sigma\|_{L^2(\Gamma)}^2 + \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)}^2 \\ &\lesssim \quad \Big\|h_\star^{1/2}\partial_\Gamma(f - V\Phi_\star) \sum_{z \in \widetilde{\mathcal{N}}_\star} \varphi_z\Big\|_{L^2(\Gamma)}^2 + \Big\|h_\star^{1/2}(f - V\Phi_\star) \sum_{z \in \widetilde{\mathcal{N}}_\star} \partial_\Gamma \varphi_z\Big\|_{L^2(\Gamma)}^2 + \|h_\star^{-1/2}\Sigma\|_{L^2(\Gamma)}^2 \\ &\lesssim \quad \|h_\star^{1/2}\partial_\Gamma(f - V\Phi_\star)\|_{L^2(\omega_\star^{p+1}([\mathcal{T}_\star] \backslash [\mathcal{T}_+]))}^2 + \|h_\star^{-1/2}(f - V\Phi_\star)\|_{L^2(\omega_\star^{p+1}([\mathcal{T}_\star] \backslash [\mathcal{T}_+]))}^2.\end{aligned} \tag{5.12}$$

It remains to consider the term $\|h_\star^{-1/2}(f - V\Phi_\star)\|_{L^2(\omega^{p+1}([\mathcal{T}_\star] \backslash [\mathcal{T}_+]))}$ of (5.11) and (5.12). It holds

$$\|h_\star^{-1/2}(f - V\Phi_\star)\|_{L^2(\omega_\star^{p+1}([\mathcal{T}_\star] \backslash [\mathcal{T}_+]))}^2 = \sum_{\substack{T \in \mathcal{T}_\star \\ T \subseteq \omega_\star^{p+1}([\mathcal{T}_\star] \backslash [\mathcal{T}_+])}} \|h_\star^{-1/2}(f - V\Phi_\star)\|_{L^2(T)}^2. \tag{5.13}$$

For any $T \in \mathcal{T}_\star$, there is a function $\psi_T \in \mathcal{X}_\star$ with connected support, $T \subseteq \operatorname{supp}(\psi_T) \subseteq \omega_\star^{\lceil p/2 \rceil}(T)$ and $\|1-\psi_T\|^2_{L^2(\operatorname{supp}(\psi_T))} \le q\,|\operatorname{supp}(\psi_T)|$ with some $q \in (0,1)$ which depends only on $\check{\kappa}_{\max}, \gamma, p, w_{\min}$, and $w_{\max}$; see [FGP15, (A1)–(A2), Theorem 4.4]. We use some Poincaré inequality (see, e.g., [Fae00, Lemma 2.5]) to see

$$\begin{aligned}
&\|f - V\Phi_\star\|^2_{L^2(\operatorname{supp}\psi_T)} \\
&\quad \le \frac{|\operatorname{supp}(\psi_T)|^2}{2} \|\partial_\Gamma (f - V\Phi_\star)\|^2_{L^2(\operatorname{supp}(\psi_T))} + \frac{1}{|\operatorname{supp}(\psi_T)|} \left| \int_{\operatorname{supp}(\psi_T)} (f - V\Phi_\star)(x)\,dx \right|^2 .
\end{aligned} \tag{5.14}$$

The Galerkin orthogonality proves

$$\begin{aligned}
\left| \int_{\operatorname{supp}(\psi_T)} (f - V\Phi_\star)(x)\,dx \right|^2 &= \left| \int_{\operatorname{supp}(\psi_T)} (f - V\Phi_\star)(x)(1-\psi_T(x))\,dx \right|^2 \\
&\le \|f - V\Phi_\star\|^2_{L^2(\operatorname{supp}(\psi_T))} q\,|\operatorname{supp}(\psi_T)|.
\end{aligned}$$

Using (5.14), we therefore get

$$\|f - V\Phi_\star\|^2_{L^2(\operatorname{supp}(\psi_T))} \le \frac{|\operatorname{supp}(\psi_T)|^2}{2} \|\partial_\Gamma (f - V\Phi_\star)\|^2_{L^2(\operatorname{supp}(\psi_T))} + q\,\|f - V\Phi_\star\|^2_{L^2(\operatorname{supp}(\psi_T))},$$

which implies

$$\|f - V\Phi_\star\|^2_{L^2(T)} \lesssim h^2_{\star,T} \|\partial_\Gamma (f - V\Phi_\star)\|^2_{L^2(\omega^{\lceil p/2 \rceil}(T))}.$$

Hence, we are led to

$$\|h_\star^{-1/2}(f - V\Phi_\star)\|^2_{L^2(\omega^{p+1}([\mathcal{T}_\star] \setminus [\mathcal{T}_+]))} \lesssim \|h_\star^{1/2} \partial_\Gamma (f - V\Phi_\star)\|^2_{L^2(\omega^{\lceil p/2 \rceil + p+1}([\mathcal{T}_\star] \setminus [\mathcal{T}_+]))}.$$

With

$$\mathcal{R}_\star(\mathcal{T}_+) := \big\{ T \in \mathcal{T}_\star \,:\, T \subseteq \omega^{\lceil p/2 \rceil + p + 1}([\mathcal{T}_\star] \setminus [\mathcal{T}_+]) \big\},$$

we therefore conclude the proof. □

Since we use a different mesh-refinement strategy, we cannot directly cite the following lemma from [CFPP14]. However, we may essentially follow the proof of [CFPP14, Proposition 4.12] verbatim. Details are left to the reader.

Lemma 5.4 (optimality of Dörfler marking). *Define*

$$\overline{\theta}_{\rm opt} := (1 + C^2_{\rm stab} C^2_{\rm rel})^{-1}. \tag{5.15}$$

For all $0 < \overline{\theta} < \overline{\theta}_{\rm opt}$ *there is some* $0 < q_{\rm opt} < 1$ *such that for all refinements* $[\mathcal{T}_+] \in$ `ref`$([\mathcal{T}_\star])$ *of* $[\mathcal{T}_\star] \in [\mathbb{T}]$ *the following implication holds true*

$$\rho_+^2 \le q_{\rm opt} \rho_\star^2 \quad \Longrightarrow \quad \overline{\theta} \rho_\star^2 \le \sum_{T \in \mathcal{R}_\star(\mathcal{T}_+)} \rho_\star(T)^2. \tag{5.16}$$

The constant $q_{\rm opt}$ *depends only on* $\overline{\theta}$*, and the constants* $C_{\rm stab}$ *and* $C_{\rm rel}$ *of Lemma 5.1 and 5.2.* □

The next lemma reads similarly as [CFPP14, Lemma 3.4]. Since we use a different mesh-refinement strategy and our estimator ρ does not satisfy the reduction axiom (A2), we cannot directly cite the result. However, the idea of the proof is the same. Indeed, one only needs a weaker version of the mentioned axiom.

Lemma 5.5 (quasi-monotonicity of ρ). *For all refinements $[\mathcal{T}_+] \in \texttt{ref}([\mathcal{T}_\star])$ of $[\mathcal{T}_\star] \in [\mathbb{T}]$, there holds*

$$\rho_+^2 \leq C_{\mathrm{mon}} \rho_\star^2, \tag{5.17}$$

where $C_{\mathrm{mon}} > 0$ depends only on the parametrisation γ and the constants C_{inv} of Proposition 4.1 and C_{rel} of Lemma 5.2.

Proof. We split the estimator as follows

$$\rho_+^2 = \sum_{T \in \mathcal{T}_+ \setminus \mathcal{T}_\star} \rho_+(T)^2 + \sum_{T \in \mathcal{T}_\star \cap \mathcal{T}_+} \rho_+(T)^2.$$

For the first sum, we use (5.4), $\bigcup(\mathcal{T}_+ \setminus \mathcal{T}_\star) = \bigcup(\mathcal{T}_\star \setminus \mathcal{T}_+)$, and $\check{h}_+ \leq \check{h}_\star$ to estimate

$$\begin{aligned}
\sum_{T \in \mathcal{T}_+ \setminus \mathcal{T}_\star} \rho_+(T)^2 &= \|\check{h}_+^{1/2} \partial_\Gamma (f - V\Phi_+)\|^2_{L^2(\bigcup(\mathcal{T}_+ \setminus \mathcal{T}_\star))} \\
&\lesssim \Big(\|\Phi_+ - \Phi_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|\check{h}_\star^{1/2} \partial_\Gamma (f - V\Phi_\star)\|_{L^2(\bigcup(\mathcal{T}_\star \setminus \mathcal{T}_+))} \Big)^2 \\
&\leq 2\|\Phi_+ - \Phi_\star\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + 2 \sum_{T \in \mathcal{T}_\star \setminus \mathcal{T}_+} \rho_\star(T)^2
\end{aligned}$$

For the second sum, we use Lemma 5.1 to see

$$\sum_{T \in \mathcal{T}_\star \cap \mathcal{T}_+} \rho_+(T)^2 \leq 2 \sum_{T \in \mathcal{T}_\star \cap \mathcal{T}_+} \rho_\star(T)^2 + 2C_{\mathrm{stab}}^2 \|\Phi_\star - \Phi_+\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.$$

We end up with

$$\rho_+^2 \lesssim \|\Phi_+ - \Phi_\star\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \rho_\star^2.$$

Lemma 5.2 concludes the proof. $\square$

The optimality in Theorem 3.2 essentially follows from the following lemma. It was inspired by an analogous version from [CFPP14, Lemma 4.14].

Lemma 5.6. *Suppose that $\phi \in \mathbb{A}_s$ for some $s > 0$. Then, for all $0 < \overline{\theta} < \overline{\theta}_{\mathrm{opt}}$ there exist constants $C_1, C_2 > 0$ such that for all meshes $[\mathcal{T}_\star] \in [\mathbb{T}]$ there exists some refinement $[\mathcal{T}_+] \in \texttt{ref}([\mathcal{T}_\star])$ such that the corresponding set $\mathcal{R}_\star(\mathcal{T}_+) \subseteq \mathcal{T}_\star$ from Lemma 5.2 satisfies*

$$|\mathcal{R}_\star(\mathcal{T}_+)| \leq C_1 C_2^{1/s} \|\phi\|_{\mathbb{A}_s}^{1/s} \rho_\star^{-1/s}, \tag{5.18}$$

and

$$\overline{\theta} \rho_\star^2 \leq \sum_{T \in \mathcal{R}_\star(\mathcal{T}_+)} \rho_\star(T)^2. \tag{5.19}$$

With the constants $C_{\mathrm{rel}}, C_{\mathrm{mon}}$, and q_{opt} from Lemma 5.2, 5.4 and 5.5, it holds $C_1 = 2C_{\mathrm{rel}}$ and $C_2 = (C_{\mathrm{mon}} q_{\mathrm{opt}}^{-1})^{1/2}$.

Proof. We set $\alpha := C_{\mathrm{mon}}^{-1} q_{\mathrm{opt}}$ with the constants of Lemma 5.4 and Lemma 5.5, and $\delta^2 := \alpha \rho_\star^2$.

Step 1: There exists $[\mathcal{T}_\delta] \in [\mathbb{T}]$ with

$$\rho_\delta \leq |\mathcal{K}_\delta| - |\mathcal{K}_0| \leq \|\phi\|_{\mathbb{A}_s}^{1/s} \delta^{-1/s}.$$

Let $N \in \mathbb{N}_0$ be minimal with $(N+1)^{-s}\|\phi\|_{\mathbb{A}_s} \le \delta$. If $N = 0$, then $\rho_0 \le \|\phi\|_{\mathbb{A}_s} \le \delta$ and we can choose $[\mathcal{T}_\delta] = [\mathcal{T}_0]$. If $N > 0$, minimality of N implies $N^{-s}\|\phi\|_{\mathbb{A}_s} > \delta$ or equivalently $N+1 \le \|\phi\|_{\mathbb{A}_s}^{1/s}\delta^{-1/s}$. Now we choose $[\mathcal{T}_\delta] \in [\mathbb{T}_N]$ such that

$$\rho_\delta = \min_{[\mathcal{T}_\star]\in[\mathbb{T}_N]} \rho_\star$$

and see

$$\rho_\delta \le (N+1)^{-s}\|\phi\|_{\mathbb{A}_s} \le \delta.$$

Step 2: We consider the overlay $[\mathcal{T}_+] := [\mathcal{T}_\star] \oplus [\mathcal{T}_\delta]$ of (M2). Quasi-monotonicity shows

$$\rho_+^2 \le C_{\rm mon}\rho_\delta^2 \le C_{\rm mon}\delta^2 = q_{\rm opt}\rho_\star^2. \tag{5.20}$$

Step 3: Finally, the assumptions on the refinement strategy are used. The overlay estimate and Step 1 give

$$|\mathcal{K}_+| - |\mathcal{K}_\star| \le (|\mathcal{K}_\delta| + |\mathcal{K}_\star| - |\mathcal{K}_0|) - |\mathcal{K}_\star| = |\mathcal{K}_\delta| - |\mathcal{K}_0| \le \|\phi\|_{\mathbb{A}_s}^{1/s}\delta^{-1/s}.$$

Lemma 5.2 and (M3) show

$$|\mathcal{R}_\star(\mathcal{T}_+)| \le C_{\rm rel}|[\mathcal{T}_\star] \setminus [\mathcal{T}_+]| \le 2C_{\rm rel}(|\mathcal{K}_+| - |\mathcal{K}_\star|).$$

Combining the last two estimates, we end up with

$$|\mathcal{R}_\star(\mathcal{T}_+)| \le 2C_{\rm rel}\|\phi\|_{\mathbb{A}_s}^{1/s}\alpha^{-1/2s}\rho_\star^{-1/s},$$

This proves (5.18) with $C_1 = 2C_{\rm rel}$ and $C_2 = \alpha^{-1/2}$. By (5.20) we can apply Lemma 5.4 and see (5.19). $\square$

So far, we have only considered the auxiliary estimator $\rho_\star$ In particualar, we did not use Algorithm 3.1, but only the refinement strategy `ref` itself. For the proof of optimal convergence (3.7), we proceed similarly as in [CFPP14, Theorem 8.4 (ii)].

Proof of (3.7). Due to (5.2), there is a constant $C \ge 1$ which depends only on $\check{\kappa}_{\max}$ with $\mu_\star^2 \le C\rho_\star^2$ for all $[\mathcal{T}_\star] \in [\mathbb{T}]$. We set $\theta_{\rm opt} := \overline{\theta}_{\rm opt}/C$ and $\overline{\theta} := C\,\theta$ and suppose that θ is sufficiently small, namely, $\theta < \theta_{\rm opt}$ and hence $\overline{\theta} < \overline{\theta}_{\rm opt}$. Let $\ell \in \mathbb{N}_0$ and $j \le \ell$. Choose a refinement $[\mathcal{T}_+]$ of $[\mathcal{T}_j]$ as in Lemma 5.6. In particular, the set $\mathcal{R}_j(\mathcal{T}_+)$ satisfies the Dörfler marking (5.20). According to (5.2), this implies

$$\theta\mu_j^2 \le \overline{\theta}\rho_j^2 \le \sum_{T\in\mathcal{R}_j(\mathcal{T}_+)} \rho_j(T)^2 \le \sum_{z\in\mathcal{N}_j\cap\bigcup\mathcal{R}_j(\mathcal{T}_+)} \mu_j(z)^2,$$

i.e., the set $\mathcal{N}_j \cap \bigcup \mathcal{R}_j(\mathcal{T}_+)$ satisfies the Dörfler marking (3.3) from Algorithm 3.1. Since the chosen set $\mathcal{M}_j$ of Algorithm 3.1 has essentially minimal cardinality, we see with (5.18) that

$$|\mathcal{M}_j| \le C_{\rm mark}|\mathcal{N}_j \cap \bigcup \mathcal{R}_j(\mathcal{T}_+)| \le 2C_{\rm mark}|\mathcal{R}_j(\mathcal{T}_+)| \le 2C_{\rm mark}C_1C_2^{1/s}\|\phi\|_{\mathbb{A}_s}^{1/s}\rho_j^{-1/s}$$

With the mesh-closure estimate of (M2), we get

$$\begin{aligned}|\mathcal{K}_\ell| - |\mathcal{K}_0| &\le C_{\rm mesh}\sum_{j=0}^{\ell-1}|\mathcal{M}_j| \le 2C_{\rm mark}C_{\rm mesh}C_1C_2^{1/s}\|\phi\|_{\mathbb{A}_s}^{1/s}\sum_{j=0}^{\ell-1}\rho_j^{-1/s}\\ &\le 2C_{\rm mark}C_{\rm mesh}C_1C_2^{1/s}C^{1/s}\|\phi\|_{\mathbb{A}_s}^{1/s}\sum_{j=0}^{\ell-1}\mu_j^{-1/s}.\end{aligned}$$

Linear convergence (3.5) shows

$$\mu_\ell \le C_{\rm lin}q_{\rm lin}^{\ell-j}\mu_j \quad \text{for all } j = 0,\dots,\ell.$$

Hence,

$$\begin{aligned}|\mathcal{K}_\ell| - |\mathcal{K}_0| &\le 2C_{\mathrm{mark}}C_{\mathrm{mesh}}C_1C_2^{1/s}C^{1/s}\|\phi\|_{\mathbb{A}_s}^{1/s}\sum_{j=0}^{\ell-1}\mu_j^{-1/s}\\ &\le 2C_{\mathrm{mark}}C_{\mathrm{mesh}}C_1(C_2C_{\mathrm{lin}}C)^{1/s}\|\phi\|_{\mathbb{A}_s}^{1/s}\mu_\ell^{-1/s}\sum_{j=0}^{\ell-1}(q_{\mathrm{lin}}^{1/s})^{\ell-j}\\ &\le (C_2C_{\mathrm{lin}}C)^{1/s}\frac{2C_{\mathrm{mark}}C_{\mathrm{mesh}}C_1}{1-q_{\mathrm{lin}}^{1/s}}\|\phi\|_{\mathbb{A}_s}^{1/s}\mu_\ell^{-1/s}.\end{aligned}$$

This concludes the proof. □

Acknowledgement. The authors acknowledge support through the Austrian Science Fund (FWF) under grant P27005 *Optimal adaptivity for BEM and FEM-BEM coupling.* In addition, DP, MF, and GG are supported through the FWF doctoral school *Nonlinear PDEs* funded under grant W1245.

References

[AFF+13] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, and Dirk Praetorius. Efficiency and optimality of some weighted-residual error estimator for adaptive 2D boundary element methods. *Comput. Methods Appl. Math.*, 13(3):305–332, 2013.

[AFF+15] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, Jens Markus Melenk, and Dirk Praetorius. Local inverse estimates for non-local boundary integral operators. *in progress*, 2015.

[BBdVC+06] Yuri Bazilevs, Lourenco Beirão da Veiga, J. Austin Cottrell, Thomas J.R. Hughes, and Giancarlo Sangalli. Isogeometric analysis: approximation, stability and error estimates for h-refined meshes. *Mathematical Models and Methods in Applied Sciences*, 16(07):1031–1090, 2006.

[BdVBSV14] Lourenco Beirão da Veiga, Annalisa Buffa, Giancarlo Sangalli, and Rafael Vázquez. Mathematical analysis of variational isogeometric methods. *Acta Numer.*, 23:157–287, 2014.

[BG15] Annalisa Buffa and Carlotta Giannelli. Adaptive isogeometric methods with hierarchical splines: error estimator and convergence. *preprint arXiv:1502.00565*, 2015.

[Car97] Carsten Carstensen. An a posteriori error estimate for a first-kind integral equation. *Math. Comp.*, 66(217):139–155, 1997.

[CFPP14] Carsten Carstensen, Michael Feischl, Marcus Page, and Dirk Praetorius. Axioms of adaptivity. *Comput. Math. Appl.*, 67(6):1195–1253, 2014.

[CHB09] J. Austin Cottrell, Thomas J.R. Hughes, and Yuri Bazilevs. *Isogeometric analysis: toward integration of CAD and FEA*. John Wiley & Sons, 2009.

[CKNS08] J. Manuel Cascon, Christian Kreuzer, Ricardo H. Nochetto, and Kunibert G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[CMPS04] Carsten Carstensen, Matthias Maischak, Dirk Praetorius, and Ernst P. Stephan. Residual-based a posteriori error estimate for hypersingular equation on surfaces. *Numer. Math.*, 97(3):397–425, 2004.

[CMS01] Carsten Carstensen, Matthias Maischak, and Ernst P. Stephan. A posteriori error estimate and h-adaptive algorithm on surfaces for Symm's integral equation. *Numer. Math.*, 90(2):197–213, 2001.

[CS96] Carsten Carstensen and Ernst P. Stephan. Adaptive boundary element methods for some first kind integral equations. *SIAM J. Numer. Anal.*, 33(6):2166–2183, 1996.

[dB86] Carl de Boor. *B (asic)-spline basics.* Mathematics Research Center, University of Wisconsin-Madison, 1986.

[DFG+04] Wolfgang Dahmen, Birgit Faermann, Ivan G. Graham, Wolfgang Hackbusch, and Stefan A. Sauter. Inverse inequalities on non-quasi-uniform meshes and application to the mortar element method. *Math. Comp.*, 73(247):1107–1138, 2004.

[DJS10] Michael R. Dörfel, Bert Jüttler, and Bernd Simeon. Adaptive isogeometric analysis by local h-refinement with T-splines. *Comput. Methods Appl. Mech. Engrg.*, 199(5-8):264–275, 2010.

[Dör96] Willy Dörfler. A convergent adaptive algorithm for poisson's equation. *SIAM Journal on Numerical Analysis*, 33(3):1106–1124, 1996.

[Fae00] Birgit Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. I. The two-dimensional case. *IMA J. Numer. Anal.*, 20(2):203–234, 2000.

[FFH+14] Michael Feischl, Thomas Führer, Norbert Heuer, Michael Karkulik, and Dirk Praetorius. Adaptive boundary element methods: A posteriori error estimators, adaptivity, convergence, and implementation. *Arch. Comput. Methods Engrg.*, published online first, 2014.

[FFK+14] Michael Feischl, Thomas Führer, Michael Karkulik, Jens Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rates for adaptive boundary element methods with data approximation, part I: weakly-singular integral equation. *Calcolo*, 51(4):531–562, 2014.

[FFK+15] Michael Feischl, Thomas Führer, Michael Karkulik, Jens Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rates for adaptive boundary element methods with data approximation, part II: hyper-singular integral equation. *Electron. Trans. Numer. Anal.*, 44:153–176, 2015.

[FFKP14] Michael Feischl, Thomas Führer, Michael Karkulik, and Dirk Praetorius. ZZ-type a posteriori error estimators for adaptive boundary element methods on a curve. *Eng. Anal. Bound. Elem.*, 38:49–60, 2014.

[FFME+14] Michael Feischl, Thomas Führer, Gregor Mitscha-Eibl, Dirk Praetorius, and Ernst P. Stephan. Convergence of adaptive BEM and adaptive FEM-BEM coupling for estimators without h-weighting factor. *Comput. Methods Appl. Math.*, 14(4):485–508, 2014.

[FGHP15] Michael Feischl, Gregor Gantner, Alexander Haberl, and Dirk Praetorius. Adaptive 2D IGA boundary element methods. *preprint arXiv:1504.06164*, 2015.

[FGP15] Michael Feischl, Gregor Gantner, and Dirk Praetorius. Reliable and efficient a posteriori error estimation for adaptive iga boundary element methods for weakly-singular integral equations. *Comput. Methods Appl. Mech. Engrg.*, 290:362–386, 2015.

[FKMP13] Michael Feischl, Michael Karkulik, Jens Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rate for an adaptive boundary element method. *SIAM J. Numer. Anal.*, 51(2):1327–1348, 2013.

[Gan13] Tsogtgerel Gantumur. Adaptive boundary element methods with convergence rates. *Numer. Math.*, 124(3):471–516, 2013.

[Gan14] Gregor Gantner. Isogeometric adaptive BEM. Master's thesis, Vienna University of Technology, 2014.

[Geo08] Emmanuil Georgoulis. Inverse-type estimates on -finite element spaces and applications. *Mathematics of Computation*, 77(261):201–219, 2008.

[GHS05] Ivan G. Graham, Wolfgang Hackbusch, and Stefan A. Sauter. Finite elements on degenerate meshes: inverse-type inequalities and applications. *IMA J. Numer. Anal.*, 25(2):379–407, 2005.

[HAD14] Luca Heltai, Marino Arroyo, and Antonio DeSimone. Nonsingular isogeometric boundary element method for Stokes flows in 3D. *Comput. Methods Appl. Mech. Engrg.*, 268:514–539, 2014.

[HCB05] Thomas J. R. Hughes, J. Austin Cottrell, and Yuri Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, 194(39-41):4135–4195, 2005.

[HW08] George C. Hsiao and Wolfgang L. Wendland. *Boundary integral equations*, volume 164 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2008.

[McL00] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[PGK+09] Costas Politis, Alexandros I. Ginnis, Panagiotis D. Kaklis, Kostas Belibassakis, and Christian Feurer. An isogeometric BEM for exterior potential-flow problems in the plane. In *2009 SIAM/ACM Joint Conference on Geometric and Physical Modeling*, pages 349–354. ACM, 2009.

[PTC13] Michael J. Peake, Jon Trevelyan, and Graham Coates. Extended isogeometric boundary element method (XIBEM) for two-dimensional Helmholtz problems. *Comput. Methods Appl. Mech. Engrg.*, 259:93–102, 2013.

[SBTR12] Robert N. Simpson, Stephane P.A. Bordas, Jon Trevelyan, and Timon Rabczuk. A two-dimensional isogeometric boundary element method for elastostatic analysis. *Comput. Methods Appl. Mech. Engrg.*, 209/212:87–100, 2012.

[SS11] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original.

[SSE+13] Michael A. Scott, Robert N. Simpson, John A. Evans, Scott Lipton, Stephane P.A. Bordas, Thomas J. R. Hughes, and Thomas W. Sederberg. Isogeometric boundary element analysis using unstructured T-splines. *Comput. Methods Appl. Mech. Engrg.*, 254:197–221, 2013.

[Ste08] Olaf Steinbach. *Numerical approximation methods for elliptic boundary value problems.* Springer, New York, 2008. Translated from the 2003 German original.

[TM12] Toru Takahashi and Toshiro Matsumoto. An application of fast multipole method to isogeometric boundary element method for Laplace equation in two dimensions. *Eng. Anal. Bound. Elem.*, 36(12):1766–1775, 2012.

[VGJS11] Anh-Vu Vuong, Carlotta Giannelli, Bert Jüttler, and Bernd Simeon. A hierarchical approach to adaptive local refinement in isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 200(49-52):3554–3567, 2011.