REMARKS ON THE CAUCHY PROBLEM FOR THE ZAKHAROV-RUBENCHIK/ BENNEY-ROSKES SYSTEM

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Dedicated to Vladimir Georgiev with admiration and frienship

Abstract. We address various issues on the Cauchy problem for the Zakharov-Rubenchik system, (known as the Benney-Roskes system in water waves theory) which models the interaction of short and long waves in many physical situations. Motivated by the transverse stability/instability of the one-dimensional solitary wave (line solitary), we study the Cauchy problem in the background of a line solitary wave.

1. Introduction

This paper is concerned with various issues concerning the Cauchy problem for the two or three-dimensional Zakharov-Rubenchik (or Benney-Roskes) system and its perturbation by a line soliton. The Zakharov-Rubenchik system is by no doubts a fundamental one, being a "generic" asymptotic system in the so-called modulation regime (slowly varying envelope of a fast oscillating train) and it was actually derived in various physical contexts. Moreover it contains in various limits the classical Zakharov system (coupling a nonlinear Schrödinger equation and a wave equation, see (1.3) below) and the Davey-Stewartson systems (coupling a nonlinear Schrödinger equation and an elliptic equation). We refer to [50] for more details on the formal derivation of those systems and on the physical background.

The Davey-Stewartson system was first derived formally in the context of water waves in [14, 1, 15] (see also [11, 12] for a derivation of Davey-Stewartson systems in a different context). However, as noticed in [25] it is less general than the

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Benney-Roskes system (1.6) below in the sense that the initial conditions have to be prepared to obtain an approximation of the full water waves system.

We refer to [13] for a rigorous justification of the Zakharov limit of the Zakharov-Rubenchik system and to [34] for the Schrödinger limit of the Zakharov-Rubenchik system in the one-dimensional case and for well-prepared initial data.

The Zakharov-Rubenchik/Benney-Roskes system is thus richer than those simpler models and should capture more of the original dynamics. It was introduced in [51] (see also the survey article [50]) to describe the interaction of spectrally narrow high-frequency wave packet of small amplitude with a low-frequency acoustic type oscillations. The analysis is general and carried out in the Hamiltonian formalism and yields the following *universal* system

$$\begin{cases} \psi_t + v_g\psi_z + i\frac{\omega"}{2}\partial_z^2\psi + i\frac{v_g}{2k}\Delta_\perp\psi - i(q|\psi|^2 + \beta\rho + \alpha\partial_z\phi)\psi = 0, \\ \rho_t + \rho_0\Delta\phi + \alpha\partial_z|\psi|^2 = 0 \\ \phi_t + \frac{c^2}{\rho_0}\rho + \beta|\psi|^2 = 0, \end{cases} \tag{1.1}$$

where the two last equations describe the acoustic type waves and $\Delta_\perp = \partial_x^2 + \partial_y^2$ or ∂_x^2, $\Delta = \Delta_\perp + \partial_z^2$.

In two space dimensions a more specific (formal) derivation in the context of surface water waves is displayed in [6] and rigorously justified in [25], see below for a more precise description.

In the notations of [39] (see also [38] where it is used in the context of Alfvén waves in dispersive MHD), the Zakharov-Rubenchik system has the form

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\delta\psi_{xx} - i\sigma_1\Delta_\perp\psi + i\left\{\sigma_2|\psi|^2 + W(\rho + D\phi_x)\right\}\psi = 0 \\ \rho_t + \Delta\phi + D|\psi|_x^2 = 0 \\ \phi_t + \frac{1}{M^2}\rho + |\psi|^2 = 0, \end{cases} \tag{1.2}$$

where $\psi : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C},\ \rho, \phi : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}, d = 2, 3$ describe respectively the fast oscillating and acoustic type waves.

Here $\sigma_1, \sigma_2, \sigma_3 = \pm 1$, $W > 0$ measures the strength of the coupling with acoustic type waves, $M > 0$ is a Mach number, $D \in \mathbb{R}$ is associated to the Doppler shift due to the medium velocity and $\delta \in \mathbb{R}$ is a nondimensional dispersion coefficient.

When $\alpha = 0$ (resp. $D = 0$) in (1.1) (resp. (1.2)) the Zakharov-Rubenchik system reduces to the classical (scalar) Zakharov system (see *eg* Chapter V in [44]). More precisely, in the framework of (1.1), one gets

$$\begin{cases} \psi_t + v_g\psi_z + i\frac{\omega"}{2}\partial_z^2\psi + i\frac{v_g}{2k}\Delta_\perp\psi = i(q|\psi|^2 + \beta\rho)\psi, \\ \rho_{tt} - c^2\Delta\rho = \beta\rho_0\Delta|\psi|^2, \end{cases} \tag{1.3}$$

which is a form of the 2 or $3D$ Zakharov system. Note however that the second order operator in the first equation is not necessarily elliptic.

The local well-posedness in $H^s(\mathbb{R}^d)\times H^{s-1/2}(\mathbb{R}^d)\times H^{s+1/2}(\mathbb{R}^d)$ with $s>\frac{d}{2}, d=2,3$ for (1.2), (1.1) was obtained in [39] by using the local smoothing property of the free Schrödinger operator after reducing the system to a quasilinear (non local) Schrödinger equation. Since it uses dispersive properties of the free Schrödinger group that are valid only in the whole space the proof does not extend to the Cauchy problem posed in $\mathbb{T}^d$ or $\mathbb{R}^{d-1}\times\mathbb{T}$, the later situation being relevant for transverse stability issues. On the other hand, when applied to the Benney-Roskes system (1.5) below it provides an existence time of order $O(1)$ [1]while an existence time of order $O(1/\epsilon)$ is needed to fully justify the Benney-Roskes as a water wave model on the correct time scales (see [24]).

Local well-posedness of the Zakharov-Rubenchik/Benney-Roskes system was also obtained in [29], for $s>2$ with the additional condition $\delta\sigma_1>0$ (that is the second order operator in the first equation of (1.2), (1.1) is elliptic) by using a energy method inspired by the work of Schochet-Weinstein in [43] on the nonlinear Schrödinger limit of the Zakharov system. The method used in [29] and [43] consists in rewriting the Zakharov system (or the Zakharov-Rubenchik system) as a dispersive (skew-adjoint) perturbation of a symmetric nonlinear hyperbolic system and it uses only the algebraic structure of the system.

We will see that, when the small parameter ϵ is included, this method provides also the existence on the time scale $O(1)$ in the context of water waves (see the Benney-Roskes system (1.6) below) and moreover that it can be applied to the system obtained from (1.2) which is satisfied by a (localized) perturbation of a line soliton. Also, since it does not use any dispersive property of the Schrödinger group, it applies to the Cauchy problem in $\mathbb{T}^d$ or $\mathbb{R}^{d-1}\times\mathbb{T}$, a situation that have not been addressed before (see on the other hand [7, 8] for the periodic Zakharov system) .

Thus, none of the two aforementioned methods seems to give the expected existence time scale for the Benney-Roskes system. Nevertheless they provide different results for Zakharov-Rubenchik type systems. The "dispersive method" used in [39] works only in $\mathbb{R}^d$ but does not need the Schrödinger part of the system to be "elliptic" (that is it does not needs the condition $\delta\sigma_1>0$). Also it lowers the regularity on the initial data (an effect of the dispersive smoothing effect) and could be applied as well to (possibly non physical) nonlinear perturbations of the system.

On the other hand, the Schochet-Weinstein type, "hyperbolic like" methods allow to deal with the periodic or semi-periodic cases, but are relatively rigid (they rely on the algebraic structure of the system) and need to take initial data in the "hyperbolic space" $H^s(\mathbb{R}^d), s>\frac{d}{2}+1$. .

The situation is better understood is spatial dimension one. Oliveira [33] proved the local (thus global using the conservation laws below) well-posedness in $H^2(\mathbb{R})\times H^1(\mathbb{R})\times H^1(\mathbb{R})$. This result was improved in [26] where in particular global well-posedness was established in the energy space $H^1(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})$.

It is worth noticing that (1.2) possesses two conserved quantities, the L^2 norm

$$\int|\psi(x,y_\perp,t)|^2=\int|\psi(x,y_\perp,0)|^2$$

[1]Roughly speaking, the idea in [39] is to reduce the system to a (nonlocal) quasilinear Schrödinger equation. When ϵ is taken into account, the crucial dispersive smoothing estimate on the Schrödinger group has a $1/\epsilon$ factor while the nonlinear term has a ϵ factor.

where $y_\perp = y$ or (y,z), and, after the change of variable $(x,t) \to (x+\sigma_3 t, t)$, the Hamiltonian

(1.4)
$$E(t) = \int_{\mathbb{R}^d} \left(\frac{\delta}{2}|\psi_x|^2 + \frac{\sigma_1}{2}|\nabla_\perp \psi|^2 + \frac{\sigma_2}{4}|\psi|^4 + \frac{W}{4M^2}\rho^2 + \frac{W}{4}|\nabla\phi|^2 + \frac{W}{2}(\rho + D\phi_x)|\psi|^2 \right) = E(0),$$

The conservation laws are used !n [39] to obtain global weak solutions under suitable assumptions on the coefficients. We will use them in Section 5 to prove the global existence of weak solutions of the systems obtained by perturbing a line (dark) soliton. Note also that the conservation laws can be used to get the global well-posedness of the Zakharov-Rubenchik, Benney-Roskes system is space dimension one (see [33]).

As aforementioned, in the context of water waves, the Zakharov-Rubenchik system is known as the Benney-Roskes sytem and it was formally derived in [6]. We follow here the notations in [25] where a rigorous derivation is performed.

$$\begin{aligned} &\mathbf{k} = |\mathbf{k}|\mathbf{e}_x, \quad \omega(\mathbf{k}) = \underline{\omega}(|\mathbf{k}|), \\ &\omega = \underline{\omega}(\mathbf{k}), \quad \omega' = \underline{\omega'}(|k|), \quad \omega'' = \underline{\omega''}(|\mathbf{k}|), \\ &\text{where} \quad \underline{\omega}(\xi) = \left(\left(g + \frac{\sigma}{\rho}|\xi|^2 \right) |\xi| \tanh(\mu|\xi|) \right)^{1/2} \end{aligned} \tag{1.5}$$

is the dispersion relation dispersion of water waves and where $|\mathbf{k}|$ is a fixed wave number, g is the gravity, $\sigma \geq 0$ is a surface tension coefficient, ρ is the density of the water and μ is the shallowness parameter (square of the typical fluid depth over a typical horizontal scale) which is large or infinite in the deep water models and $\alpha = -\frac{9}{8\sigma^2}(1-\sigma^2)^2$.

The small parameter ϵ is the *wave steepness* that is the ratio of a typical amplitude of the wave over a typical horizontal scale. Recall ([25]) that the typical time scale for the solutions of (1.6) below is $1/\epsilon$ and so it is crucial to establish the well-posedness on those time scales.

The Benney-Roskes equations can then be written in dimension 2 as follows

$$\begin{cases} \partial_t \psi_{01} + \omega' \partial_x \psi_{01} - i\epsilon \frac{1}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi_{01} \\ \qquad + \epsilon i \left(|\mathbf{k}|\partial_x \psi_{00} + \frac{|\mathbf{k}|^2}{2\omega}(1-\sigma^2)\zeta_{10} + 2\frac{|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi_{01}|^2 \right) \psi_{01} = 0 \\ \partial_t \zeta_{10} + \sqrt{\mu}\Delta\psi_{00} = -2\omega|\mathbf{k}|\partial_x |\psi_{01}|^2 \\ \partial_t \psi_{00} + \zeta_{10} = -|\mathbf{k}|^2(1-\sigma^2)|\psi_{01}|^2. \end{cases} \tag{1.6}$$

It is known (see *eg* [25] Chapter 8) that $\omega' > 0$ while for purely gravity waves $(\sigma = 0)$ ω is a concave function, thus $\omega'' < 0$ and the Schrödinger equation in the Benney-Roskes system is "non elliptic".

On the other hand, in presence of surface tension, the condition $\omega'' > 0$ is possible as shown in the following computation.

For simplicity of notations, we will consider ω of the following form instead of (1.5)

$$\omega(r) = \big((1+\gamma r^2) r \tanh(\mu r)\big)^{1/2}$$

with $\gamma > 0$ depends on $(\,g,\,\rho)$, is proportional to σ and $r = |\mathbf{k}|$. We have

$$\begin{aligned}\omega'(r) &= \big((1+3\gamma r^2)\tanh(\mu r) + \mu(r+\gamma r^3)\,\mathrm{sech}^2(\mu r)\big)\\ &\quad\times \frac{1}{2}\left((r+\gamma r^3)\tanh(\mu r)\right)^{-1/2}\end{aligned}$$

and

$$\begin{aligned}&\omega''(r)\\ &\quad= -\frac{1}{4}\left((1+3\gamma r^2)\tanh(\mu r) + \mu(r+\gamma r^3)\mathrm{sech}^2(\mu r)\right)^2\,\left((r+\gamma r^3)\tanh(\mu r)\right)^{-3/2}\\ &\qquad+\frac{1}{2}\left((r+\gamma r^3)\tanh(\mu r)\right)^{-1/2}\,\bigg(6\gamma r\tanh(\mu r) + 2\mu(1+3\gamma r^2)\mathrm{sech}^2(\mu r)\\ &\qquad - 2\mu^2(r+\gamma r^3)\tanh(\mu r)\mathrm{sech}^2(\mu r)\bigg).\end{aligned}$$

We see that $\omega'(r) > 0$ with $\gamma, r > 0$, we thus will look for r such that with fixed γ, $\omega''(r) > 0$.
We assume that $\mu r \gg 1$ implying that $\mathrm{sech}(\mu r) \approx 0$ and $\tanh(\mu r) \approx 1$. Therefore we only need to choose r large enough so that

$$12\gamma r > \frac{(1+3\gamma r^2)^2}{r+\gamma r^3}$$

or

$$3\gamma^2 r^4 + 6\gamma r^2 > 1.$$

In order to apply Schochet-Weinstein method we will need the condition $\delta\sigma_1 > 0$ and we will only consider the Zakharov-Rubenchik (or Benney-Roskes) system of the form of (1.2) satisfying this condition.

The one -dimensional Zakharov-system possesses solitary wave solutions and in [33], Serra de Oliveira proved their orbital stability. One motivation of the present paper was the study of their *transverse* stability. The transverse instability of the line solitary wave for some two dimensional models such as the nonlinear Schrödinger equation (NLS), the Kadomtsev-Petviashvili equation (KP) and some general "abstract" Hamiltonian systems have been carried out extensively in [40, 41, 42].
It is thus of interest to study the transverse stability of the line soliton for the two dimensional model (1.2) and the first step is to study the Cauchy problem of a localized perturbation of (1.2) by a line soliton. Another possibility is to consider y or (y, z) - periodic perturbations of the line solitary wave, a first step being to establish the well-posedness of the Cauchy problem for the Zakharov-Rubenchik, Benney-Roskes system in $\mathbb{R}^d \times \mathbb{T},\ d = 1, 2$, which could not result from the methods used in [39] but we that achieve here and also in the pure periodic case $\mathbb{T}^{d+1}$.

In order to unify the notation we will rewrite the Benney-Roskes system (1.6) in the form of (1.2). We replace $(\psi_{01}, \frac{\zeta_{10}}{|\mathbf{k}|^2(1-\sigma^2)\sqrt{\mu}}, \frac{\psi_{00}}{|\mathbf{k}|^2(1-\sigma^2)})$ by (ψ, ρ, ϕ) and after calculating the corresponding coefficients will be as follows:

$$\begin{cases} \sigma_3 = -\omega',\, \delta = \dfrac{\epsilon\omega''}{2},\, \sigma_1 = \dfrac{\epsilon\omega'}{2|\mathbf{k}|},\, \sigma_2 = \dfrac{2\epsilon|\mathbf{k}|^4(1-\alpha)}{\omega} \\ W = \dfrac{\epsilon|\mathbf{k}|^4(1-\sigma^2)^2\sqrt{\mu}}{2\omega},\, D = \dfrac{2\omega}{|\mathbf{k}|(1-\sigma^2)\sqrt{\mu}},\, M = \mu^{-1/4}. \end{cases} \tag{1.7}$$

The paper is organized as follows. In the next section we reformulate the existence of one-dimensional solitary waves (bright and dark) in our framework. In section 3 we use the Schochet-Weinstein method to prove a local existence for the Benney-Roskes/ Zakharov-Rubenchik system, keeping the small parameter ϵ which is relevant for deep water waves. In section 4 we consider the case of a localized perturbation of a line solitary wave. Finally we prove in Section 5 the global existence of weak solutions perturbing a dark solitary wave.

We conclude the paper by a list of open questions.

Notations. We will denote $|\cdot|_p$ the norm in the Lebesgue space $L^p(\mathbb{R}),\ 1 \le p \le \infty$ and $\|\cdot\|_s$ the norm in the Sobolev space $H^s(\mathbb{R}^d),\ s \in \mathbb{R}$. $(\cdot|\cdot)_2$ denotes the scalar product in L^2. We will denote $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of a tempered distribution f. For any $s \in \mathbb{R}$, we define $|D|^s f$ by its Fourier transform $\widehat{|D|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$. We also denote $|D_x|^s f = \mathcal{F}^{-1}(|\xi_1|\hat{f})$ and $|D_y|^s f = \mathcal{F}^{-1}(|\xi_2|\hat{f})$. Finally we will denote $\Lambda = (I-\Delta)^{1/2}$ and $J_\epsilon = (I-\epsilon\Delta)^{1/2}$.

2. Existence of one dimensional solitary waves

In this section, we reframe the proof of the existence of 1-d solitary waves in [33] in our setting. The 1-d Zakharov-Rubenchik system has the form

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\delta\psi_{xx} + i\left\{\sigma_2|\psi|^2 + W(\rho + D\phi_x)\right\}\psi = 0 \\ \rho_t + \phi_{xx} + D|\psi|^2_x = 0 \\ \phi_t + \dfrac{1}{M^2}\rho + |\psi|^2 = 0. \end{cases} \tag{2.1}$$

Setting $\tilde{\phi}(x,t) = \phi_x$, (2.1) becomes

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\delta\psi_{xx} + i\left\{\sigma_2|\psi|^2 + W(\rho + D\tilde{\phi})\right\}\psi = 0 \\ \rho_t + \tilde{\phi}_x + D|\psi|^2_x = 0 \\ \tilde{\phi}_t + \dfrac{1}{M^2}\rho_x + |\psi|^2_x = 0. \end{cases} \tag{2.2}$$

Let $c \ge 0$, we look for solutions of the system (2.2) of the form

$$(e^{i\lambda t}K(x-ct),\, a|K(x-ct)|^2,\, b|K(x-ct)|^2).$$

From the last two equation of (2.2) we deduce that

$$a = \frac{-(1+cD)}{1/M^2 - c^2} \quad \text{and} \quad b = \frac{-(c + D/M^2)}{1/M^2 - c^2}. \tag{2.3}$$

Then the first equation of (2.2) is equivalent to

$$\delta \ddot{K} - i(c+\sigma_3)\dot{K} - \lambda K = (\sigma_2 + W(a+bD))\,|K|^2 K.$$

Set

$$R(x) = e^{-i(c+\sigma_3)x/2\delta} K(x)$$

then

$$\delta \ddot{R} + \left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda\right) R = (\sigma_2 + W(a+bD))\,|R|^2 R. \tag{2.4}$$

The equation (2.4) has a unique positive solution if :

$$\begin{cases} \dfrac{1}{\delta}\left(\dfrac{(c+\sigma_3)^2}{4\delta} - \lambda\right) < 0 \\ \dfrac{1}{\delta}\left(\sigma_2 + W(a+bD)\right) < 0 \end{cases}$$

or equivalently

$$\begin{cases} \dfrac{1}{\delta}\left(\dfrac{(c+\sigma_3)^2}{4\delta} - \lambda\right) < 0 \\ \dfrac{1}{\delta}\left(\sigma_2 - \dfrac{W(1+D^2/M^2+2cD)}{1/M^2-c^2}\right) < 0 \end{cases} \tag{2.5}$$

We see that if $c \to (1/M)^-$ and λ is large enough then (2.5) holds assuming that $W > 0$ and $\delta > 0$ which holds true in both models (1.2) and (1.6) .
In this case,

$$R(x) = \sqrt{\frac{2}{\sigma_2 + W(a+bD)}\left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda\right)}\operatorname{sech}\left(\sqrt{-\frac{1}{\delta}\left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda\right)}\,x\right) \tag{2.6}$$

Otherwise, if

$$\begin{cases} \dfrac{1}{\delta}\left(\dfrac{(c+\sigma_3)^2}{4\delta} - \lambda\right) > 0 \\ \dfrac{1}{\delta}\left(\sigma_2 + W(a+bD)\right) > 0 \end{cases}$$

or equivalently

$$\begin{cases} \dfrac{1}{\delta}\left(\dfrac{(c+\sigma_3)^2}{4\delta} - \lambda\right) > 0 \\ \dfrac{1}{\delta}\left(\sigma_2 - \dfrac{W(1+D^2/M^2+2cD)}{1/M^2-c^2}\right) > 0 \end{cases} \tag{2.7}$$

In the context of water waves (Benney-Roskes system) there is a regime where the condition (2.7) holds. In particular, if we choose $c > \dfrac{1}{M}$ and $\lambda < \dfrac{(c+\sigma_3)^2}{4\delta}$ then (2.7) holds, since from (1.7) we know that δ, σ_2, W and D are positive. If $c = 0$ then (2.7) is equivalent to

$$\begin{cases} \dfrac{\sigma_3^2}{4\delta} > \lambda \\ \dfrac{2|\mathbf{k}|^4(1-\alpha)}{\omega} - \dfrac{|\mathbf{k}|^4(1-\sigma^2)^2}{2\omega} - \dfrac{2\omega|\mathbf{k}^2|}{\sqrt{\mu}} > 0 \end{cases}$$

Since $\alpha < 0$ one has $\dfrac{2|\mathbf{k}|^4(1-\alpha)}{\omega} > \dfrac{|\mathbf{k}|^4(1-\sigma^2)^2}{2\omega}$. Therefore if μ is large enough (which occurs in the context of deep water waves) then the above conditions hold.

In this case,

(2.8)
$$R(x) = \sqrt{\frac{1}{\sqrt{\sigma_2 + W(a+bD)}}\left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda\right)}\tanh(\sqrt{\frac{1}{\delta}\left(\frac{(c+\sigma_3)^2}{4\delta} - \lambda\right)}\,x)$$

Then the system (2.2) has two kind of solitary waves corresponding to the two conditions (2.5) and (2.7):

$$(e^{i\lambda t}e^{i(c+\sigma_3)x/2\delta}R(x-ct),\, a\,R^2(x-ct),\, b\,R^2(x-ct))$$

Recalling that $\tilde{\phi} = \phi_x$, the solutions of system (2.1) should have thus the form

$$Q = (e^{i\lambda t}e^{i(c+\sigma_3)x/2\delta}R(x-ct),\, a\,R^2(x-ct),\, b\int R^2(x-ct)\,dx). \tag{2.9}$$

Remark: Similarly to the case of the cubic nonlinear Schrödinger equation, we will call the 1-d solitary wave corresponding to the condition (2.5) and (2.7) the "bright", "dark" soliton respectively.

3. The Z-R/B-R system

As aforementioned the asymptotic model (1.6) is a good approximation of the full water wave system on a time scale $O(1/\epsilon)$ (see [25] page 233). It is thus crucial to prove the well-posedness of the Cauchy problem on time scales of order $1/\epsilon$. However, the existence time obtained by using the method in [39] does not reach the $O(1/\epsilon)$ time scale (as we already mentionned it is of order $O(1)$)

In this section, we reproduce the proof of the local well-posedness for (1.2) by using Schochet-Weinstein method in [29] but keeping the parameter ϵ in (1.6) to estimate the existence time obtained by this method. It turns out unfortunately that one does not improve upon the previously known $O(1)$ result (see however the comments in the Introduction).

Therefore we consider the following system

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\epsilon\delta\psi_{xx} - i\epsilon\sigma_1\psi_{yy} + i\epsilon\left\{\sigma_2|\psi|^2 + W(\rho + D\phi_x)\right\}\psi = 0 \\ \rho_t + \Delta\phi + D|\psi|^2_x = 0 \\ \phi_t + \dfrac{1}{M^2}\rho + |\psi|^2 = 0, \end{cases} \tag{3.1}$$

for which we obtain a local existence result :

Theorem 3.1. *Let $\delta\sigma_1 > 0$, $s > 2$. Let*

$$\psi(\cdot,0) = \psi_0 \in H^{s+1}(\mathbb{R}^2); \quad \rho(\cdot,0) = \rho_0 \in H^s(\mathbb{R}^2); \quad \phi(\cdot,0) = \phi_0 \in H^{s+1}(\mathbb{R}^2),$$
$$WM^2(\rho_t + D\phi_{xt} + 2D|\psi|^2_x)(\cdot,0) = \nabla.V_0,\ V_0 \in H^s(\mathbb{R}^2).$$

Then there exists $T > 0$, independent on ϵ, and a unique solution $(\psi,\rho,\phi) \in C([0,T];H^{s+1}(\mathbb{R}^2)) \times C([0,T];H^s(\mathbb{R}^2)) \times C([0,T];H^{s+1}(\mathbb{R}^2))$ of (3.1)

Remark 3.1. *With some minor changes, one obtains the same result in the three-dimensional case, that is ψ_{yy} replaced by $\nabla_\perp\psi$.*

Remark 3.2. *The above theorem and its proof are valid mutatis mutandi in a periodic ($\mathbb{T}^d$)or semi-periodic ($\mathbb{R}^{d-1}\times\mathbb{T}$) setting.*

Proof. We follow closely the proof in [29], Section 3.3 but we keep track of the parameter ϵ.

We first rewrite (3.1) as a dispersive perturbation of a symmetric hyperbolic system. We take the t derivative of the second and the third equation of the system (1.2). This allows to decouple the linear parts of those equations

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\epsilon\delta\psi_{xx} - i\epsilon\sigma_1\psi_{yy} + i\epsilon\left(\sigma_2|\psi|^2 + W(\rho + D\phi_x)\right)\psi = 0 \\ \rho_{tt} - \Delta\left(\dfrac{1}{M^2}\rho + |\psi|^2\right) + D|\psi|^2_{xt} = 0 \\ \phi_{tt} - \dfrac{1}{M^2}\left(\Delta\phi + D|\psi|^2_x\right) + |\psi|^2_t = 0. \end{cases} \tag{3.2}$$

We set $\mathcal{U} = W\rho + WD\phi_x$.

We then get a coupled system for ψ and $\mathcal{U}$

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\epsilon\delta\psi_{xx} - i\epsilon\sigma_1\psi_{yy} + i\epsilon\left(\sigma_2|\psi|^2 + \mathcal{U}\right)\psi = 0 \\ \mathcal{U}_{tt} - \dfrac{1}{M^2}\Delta\mathcal{U} - W\Delta|\psi|^2 + 2\,DW|\psi|^2_{xt} - \dfrac{WD^2}{M^2}|\psi|^2_{xx} = 0. \end{cases} \tag{3.3}$$

We define an auxiliary (vector valued) function V by

$$\partial_t V = \frac{1}{M}\nabla\mathcal{U} + WM\nabla|\psi|^2 + \left(\frac{WD^2}{M}|\psi|^2_x, 0\right)^T,$$

and we consider the equivalent first order system

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\epsilon\delta\psi_{xx} - i\epsilon\sigma_1\psi_{yy} + i\epsilon\left(\sigma_2|\psi|^2 + \mathcal{U}\right) = 0 \\ \mathcal{U}_t - \dfrac{1}{M^2}\nabla.V + 2WD|\psi|^2_x = 0 \\ \partial_t V - \dfrac{1}{M^2}\nabla\mathcal{U} - WM\nabla|\psi|^2 - \left(\dfrac{WD^2}{M}|\psi|^2_x, 0\right)^T = 0. \end{cases} \tag{3.4}$$

Note that the last two equations of this system are symmetric hyperbolic with respect to $\mathcal{U}$ and V. We now set

$$\mathcal{D} = \mathcal{U} + \frac{\sigma_1}{2}|\psi|^2$$

then the system (3.4) becomes

$$\begin{cases} \psi_t - \sigma_3\psi_x - i\epsilon\delta\psi_{xx} - i\epsilon\sigma_1\psi_{yy} + i\epsilon\left((\sigma_2 - \dfrac{\sigma_1}{2})|\psi|^2 + \mathcal{D}\right)\psi = 0 \\ \mathcal{D}_t - \dfrac{1}{M^2}\nabla.V - \dfrac{\sigma_1}{2}\left(|\psi|^2_t - \sigma_3|\psi|^2_x\right) + \left(2WD - \dfrac{\sigma_1\sigma_3}{2}\right)|\psi|^2_x = 0 \\ \partial_t V - \dfrac{1}{M^2}\nabla\mathcal{D} - \left(\dfrac{2W(D^2+M^2) - \sigma_1}{2M}|\psi|^2_x, \dfrac{2WM^2 - \sigma_1}{2M}|\psi|^2_y\right) = 0. \end{cases} \tag{3.5}$$

For simplicity of notation we set

$$c_1 = 2WD - \frac{\sigma_1\sigma_3}{2},\ c_2 = \frac{2W(D^2+M^2) - \sigma_1}{2M},\ c_3 = \frac{2WM^2 - \sigma_1}{2M}.$$

Furthermore, let

$$\psi = F + iG$$

and

$$\nabla\psi = H + iL = (H_1, H_2)^T + i(L_1, L_2)^T.$$

Multiplying the first equation of (3.5) by $\bar{\psi}$ and taking the real part, we deduce that

$$\begin{aligned} |\psi|_t^2 - \sigma_3|\psi|_x^2 &= i\epsilon\delta\psi_{xx}\bar{\psi} - i\epsilon\delta\bar{\psi}_{xx}\psi + i\epsilon\sigma_1\psi_{yy}\bar{\psi} - i\epsilon\sigma_1\bar{\psi}_{yy}\psi \\ &= 2\epsilon\delta(G\partial_x H_1 - F\partial_x L_1) + 2\epsilon\sigma_1(G\partial_y H_2 - F\partial_y L_2). \end{aligned} \tag{3.6}$$

We insert (3.6) into (3.5) and then separate the real and imaginary parts of the first equation of the system. We furthermore apply the spatial gradient to the first equation of (3.5) to get the equations satisfied by H and L. This leads to the following system

$$\begin{cases} H_t - \sigma_3 H_x + \epsilon\delta L_{xx} + \epsilon\sigma_1 L_{yy} - \epsilon G\nabla\mathcal{D} \\ \qquad - \epsilon\left(\mathcal{D}L + \left(\sigma_2 - \dfrac{\sigma_1}{2}\right)\left((F^2+G^2)L + 2G(FH+GL)\right)\right) = 0 \\ L_t - \sigma_3 L_x - \epsilon\delta H_{xx} - \epsilon\sigma_1 H_{yy} + \epsilon F\nabla\mathcal{D} \\ \qquad + \epsilon\left(\mathcal{D}H + \left(\sigma_2 - \dfrac{\sigma_1}{2}\right)\left((F^2+G^2)H + 2F(FH+GL)\right)\right) = 0 \\ F_t - \sigma_3 F_x + \epsilon\delta G_{xx} + \epsilon\sigma_1 G_{yy} - \epsilon\left(\left(\sigma_2 - \dfrac{\sigma_1}{2}\right)(F^2+G^2) + \mathcal{D}\right)G = 0 \\ G_t - \rho_3 G_x - \epsilon\delta F_{xx} - \epsilon\sigma_1 F_{yy} + \epsilon\left(\left(\sigma_2 - \dfrac{\sigma_1}{2}\right)(F^2+G^2) + \mathcal{D}\right)F = 0 \\ \mathcal{D}_t - \dfrac{1}{M^2}\nabla.V - \epsilon\sigma_1\delta(G\partial_x H_1 - F\partial_x L_1) - \epsilon\sigma_1^2(G\partial_y H_2 - F\partial_y L_2) \\ \qquad + 2c_1(H_1F + L_1G) = 0 \\ \partial_t V - \dfrac{1}{M^2}\nabla\mathcal{D} - 2\left(c_2(H_1F + L_1G),\, c_3(H_2F + L_2G)\right)^T = 0. \end{cases} \tag{3.7}$$

Since $\sigma_1\delta > 0$, we can perform the following change of variables

$$H^* = (\sqrt{\delta\sigma_1}H_1, \sigma_1 H_2) \text{ and } L^* = (\sqrt{\delta\sigma_1}L_1, \sigma_1 L_2),$$

and we then set $U = (H^*, L^*, F, G, \mathcal{D}, V)$.

Therefore, (3.1) is rewritten as a dispersive perturbation of a symmetric hyperbolic given by

$$\begin{aligned} U_t + (\epsilon A_1(U) + B_1)\,U_x + (\epsilon A_2(U) + B_2)U_y + C(U)U \\ = -K_1U_{xx} - K_2U_{yy}. \end{aligned} \tag{3.8}$$

Where A_1, A_2, B_1 and B_2 are symmetric matrices, K_1, K_2 are skew symmetric matrices.

$$A_1(U) = \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & M_1(U)^T & 0_{3\times2} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times1} & 0_{3\times2} \\ M_1(U) & 0_{1\times3} & 0 & 0_{1\times2} \\ 0_{2\times3} & 0_{2\times3} & 0_{2\times1} & 0_{2\times2} \end{pmatrix},$$

with

$$M_1(U) = (-\sqrt{\delta\sigma_1}G, 0, \sqrt{\delta\sigma_1}F).$$

$$A_2(U) = \begin{pmatrix} 0 & 0_{1\times3} & 0_{1\times2} & 0 & 0_{1\times2} \\ 0_{3\times1} & 0_{3\times3} & 0_{3\times2} & M_2(U)^T & 0_{3\times2} \\ 0_{2\times1} & 0_{2\times3} & 0_{2\times2} & 0_{2\times1} & 0_{2\times2} \\ 0 & M_2(U) & 0_{1\times2} & 0 & 0_{1\times2} \\ 0_{2\times1} & 0_{2\times3} & 0_{2\times2} & 0_{2\times1} & 0_{2\times2} \end{pmatrix}$$

with

$$M_2(U) = (-\sigma_1 G, 0, \sigma_1 F).$$

The rest of the proof proceeds via a classical iteration scheme for symmetric hyperbolic system (see [28, 23]). We first regularize the initial data by taking a family of self-adjoint regularization operators J_ε as follows.
Let $j \in C_0^\infty(\mathbb{R}^2)$, *supp* $j \subset \{X = (x,y) \in \mathbb{R}^2; |X| < 1\}$, $\int j = 1$ and set $j_\varepsilon = \varepsilon^{-2} j(X/\varepsilon)$. Then, we define $J_\varepsilon u \in C^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$ by

$$J_\varepsilon u = j_\varepsilon * u.$$

Set $\varepsilon_k = 2^{-k}$, $U_0^k = J_{\varepsilon_k} U_0$. We construct local solution of (3.8) by considering the iteration scheme :

$$\begin{cases} U^0(x,y,t) = U_0^0(x,y) \\ U_t^{k+1} + \left(\epsilon A_1(U^k) + B_1\right) U_x^{k+1} + \left(A_2(U^k) + B_2\right) U_y^{k+1} + C(U^k) U^{k+1} \\ \qquad = -K_1 U_{xx}^{k+1} - K_2 U_{yy}^{k+1} \\ U^{k+1}(x,y,0) = U_0^{k+1}(x,y). \end{cases} \tag{3.9}$$

By denoting that $J^s = (1-\Delta)^{s/2}$ the Bessel potential and $V = J^s U$ then applying J^s to the second equation of (3.9) we get

$$\begin{aligned} V_t^{k+1} &+ J^s \left((\epsilon A_1(U^k) + B_1) U_x^{k+1}\right) + J^s \left(\epsilon (A_2(U^k) + B_2) U_y^{k+1}\right) + J^s \left(C(U^k) U^{k+1}\right) \\ &= -K_1 V_{xx}^{k+1} - K_2 V_{yy}^{k+1}, \end{aligned}$$

or

$$\begin{aligned} V_t^{k+1} &+ \epsilon A_1(U^k) V_x^{k+1} + \left[J^s, \epsilon A_1(U^k)\right] U_x^{k+1} + J^s(B_1 U_x^{k+1}) \\ &\quad + \epsilon A_2(U^k) V_y^{k+1} + \left[J^s, \epsilon A_2(U^k)\right] U_y^{k+1} + J^s(B_2 U_y^{k+1}) \\ &+ C(U^k) V^{k+1} + \left[J^s, C(U^k)\right] U^{k+1} \\ &= -K_1 V_{xx}^{k+1} - K_2 V_{yy}^{k+1}, \end{aligned}$$

with usual notation for commutators, $[F,G] = FG - GF$.
We multiply this equation by V^{k+1} and integrate it in $\mathbb{R}^2$, by using integration by part, the symmetry of A_j, B_j, the skew symmetry of K_j and note that B_j are constant matrices we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \left\|V^{k+1}\right\|_{L^2}^2 \\ &\quad - \frac{1}{2}\epsilon \left\langle A_1(U_x^k) V^{k+1}, V^{k+1}\right\rangle + \epsilon \left\langle \left[J^s, A_1(U^k)\right] U_x^{k+1}, V^{k+1}\right\rangle \\ &\quad - \frac{1}{2}\epsilon \left\langle A_2(U_y^k) V^{k+1}, V^{k+1}\right\rangle + \epsilon \left\langle \left[J^s, A_2(U^k)\right] U_y^{k+1}, V^{k+1}\right\rangle \\ &\quad + \left\langle C(U^k) V^{k+1}, V^{k+1}\right\rangle + \left\langle \left[J^s, C(U^k)\right] U^{k+1}, V^{k+1}\right\rangle = 0, \end{aligned} \tag{3.10}$$

with $\langle . \rangle$ denoting the L^2 scalar product. We now use the commutator estimate of Kato and Ponce (see [21])

Lemma 3.1. *Let $s \geq 0$, $f \in H^s \cap W^{1,\infty}$ and $g \in H^{s-1} \cap L^\infty$, then*

$$\|[J^s, f] g\|_{L^2} \leq C \left(\|f\|_{H^s} \|g\|_{L^\infty} + \|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}}\right). \tag{3.11}$$

Using estimate (3.11), Sobolev embedding theorem for $s > 2$ and noticing that $A_j(U)$ depend linearly on U, $C(U)$ is quadratic, and the orders of ϵ in $C(U)$ are one and zero, we obtain

$$\begin{aligned}
&-\frac{1}{2}\epsilon\left\langle A_1(U^k_x)V^{k+1},V^{k+1}\right\rangle+\epsilon\left\langle\left[J^s,A_1(U^k)\right]U^{k+1}_x,V^{k+1}\right\rangle\\
&-\frac{1}{2}\epsilon\left\langle A_2(U^k_y)V^{k+1},V^{k+1}\right\rangle+\epsilon\left\langle\left[J^s,A_2(U^k)\right]U^{k+1}_y,V^{k+1}\right\rangle\\
&+\left\langle C(U^k)V^{k+1},V^{k+1}\right\rangle+\left\langle\left[J^s,C(U^k)\right]U^{k+1},V^{k+1}\right\rangle\\
&\le C\bigg(\epsilon\left\|U^k_x\right\|_{L^\infty}+\epsilon\left\|U^k\right\|_{H^s}+\epsilon\left\|U^k_y\right\|_{L^\infty}+\epsilon\left\|U^k\right\|^2_{L^\infty}+\left\|U^k\right\|_{L^\infty}\\
&\quad+\epsilon\left\|U^k\right\|_{H^s}\left\|U^k\right\|_{L^\infty}+\left\|U^k\right\|_{H^s}\bigg)\times\left\|V^{k+1}\right\|_{L^2}.
\end{aligned}$$

Or

$$\partial_t\left\|V^{k+1}\right\|^2_{L^2}\le C\left(\epsilon\left\|U^k\right\|^2_{H^s}+(\epsilon+1)\left\|U^k\right\|_{H^s}\right)\left\|V^{k+1}\right\|^2_{L^2}.$$

Then by Gronwall inequality we have

$$\left\|U^{k+1}\right\|^2_{H^s}\le\left\|U^{k+1}_0\right\|^2_{H^s}\exp\left(\int_0^t Cdt'\left((\epsilon+1)\left\|U^k\right\|^2_{H^s}+\epsilon\left\|U^k\right\|_{H^s}\right)\right) \tag{3.12}$$

We now consider a closed ball in $(H^s(\mathbb{R}^2))^9$

$$\mathcal{B}=\{U\in(H^s(\mathbb{R}^2))^9:\ \|U\|^2_{H^s}\le\|U_0\|^2_{H^s}+1\}.$$

Therefore by combining (3.12) and induction method we obtain that

$$\left\|U^k(t)\right\|^2_{H^s}\le\|U_0\|^2_{H^s}+1, \tag{3.13}$$

for $t\in[0,T]$ and $T=T(\epsilon,\|U_0\|_{H^s})=O(1)$.
The second equation of (3.9) then give us the uniform bound of $\left\|U^k_t\right\|_{L^\infty([0,T];(H^{s-2})^9)}$. The sequence $\{U^k\}$ is uniformly bounded in $L^\infty([0,T],(H^s)^9)$ then there exists a subsequence $\{U^{k_n}\}$ which converges to $U\in L^\infty([0,T],(H^s)^9)$ weak star. Furthermore, by Aubin-Lions lemma, there is a subsequence of $\{U^{k_n}\}$ (with the same notation) converging strongly in $L^\infty([0,T];(H^{s-2}_{loc})^9)$ and therefore in
$L^\infty([0,T];(H^{s-\delta}_{loc})^9)$ with $0<\delta<2$. This allows to pass to the limit in the nonlinear terms of (3.8). □

4. The perturbed Z-R/B-R system

In this section, we consider the Cauchy problem for (1.2) when it is perturbed by the line solitary wave Q given by (2.9). We will not expand the perturbed system but replace (ψ,ρ,ϕ) in (1.2) by $(\psi+\phi_1,\rho+\phi_2,\phi+\phi_3)$, where $Q=(\phi_1,\phi_2,\phi_3)$. Because Q is a 1-d solution of (1.2) then by using the calculation in the previous Section, we obtain that $(H_r,L_r,F_r,G_r,D_r,V_r)$ is a solution of (3.7) with $\epsilon=1$, where

$$\begin{cases}
H_r=\nabla(Re\ \phi_1),\quad L_r=\nabla(Im\ \phi_1),\\
F_r=Re\ \phi_1,\quad G_r=Im\ \phi_1,\\
\mathcal{D}_r=\mathcal{U}_r+\dfrac{\sigma_1}{2}|A_r|^2\text{ with }\mathcal{U}_r=W\phi_2+WD(\phi_3)_x,\\
\partial_tV_r-\dfrac{1}{M}\nabla\mathcal{U}_r-WM\nabla|\phi_1|^2-\left(\dfrac{WD^2}{M}|\phi_1|^2_x,0\right)^T=0.
\end{cases} \tag{4.1}$$

Similarly, $(\tilde{H},\tilde{L},\tilde{F},\tilde{G},\tilde{\mathcal{D}},\tilde{V})$ is a solution of (3.7) with $\epsilon=1$, where

$$\begin{cases} \tilde{H}=\nabla(Re(\psi+\phi_1)), \quad \tilde{L}=\nabla(Im(\psi+\phi_1)),\\ \tilde{F}=Re(\psi+\phi_1), \quad \tilde{G}=Im(\psi+\phi_1),\\ \tilde{\mathcal{D}}=\tilde{\mathcal{U}}+\dfrac{\rho_1}{2}|\psi+\phi_1|^2 \text{ with } \tilde{\mathcal{U}}=W(\rho+\phi_2)+WD(\phi+\phi_3)_x,\\ \partial_t\tilde{V}-\dfrac{1}{M}\nabla\tilde{\mathcal{U}}-WM\nabla|\psi+\phi_1|^2-\left(\dfrac{WD^2}{M}|\psi+\phi_1|_x^2,0\right)^T=0. \end{cases} \tag{4.2}$$

We now set

$$\begin{cases} H=\nabla(Re\ \psi), \quad L=\nabla(Im\ \psi),\\ F=Re\psi, \quad G=Im\psi,\\ \mathcal{D}=\mathcal{U}+\dfrac{\sigma_1}{2}(|\psi|^2+2Re(\phi_1\bar{\psi})) \text{ with } \mathcal{U}=W\rho+WD\phi_x,\\ \partial_t V-\dfrac{1}{M}\nabla\mathcal{U}-WM\nabla(|\psi|^2+2Re(\phi_1\bar{\psi}))-\left(\dfrac{WD^2}{M}(|\psi|^2+2Re(\phi_1\bar{\psi}))_x,0\right)^T=0. \end{cases} \tag{4.3}$$

Combining (3.7), (4.1) and (4.2), it transpires that $(H,L,F,G,\mathcal{D},V)$ is a solution of

$$\begin{cases} H_t-\sigma_3H_x+\delta L_{xx}+\sigma_1L_{yy}-(G+G_r)\nabla\mathcal{D}+\mathcal{R}_1=0\\ L_t-\sigma_3L_x-\delta H_{xx}-\sigma_1H_{yy}+(F+F_r)\nabla\mathcal{D}+\mathcal{R}_2=0\\ F_t-\sigma_3F_x+\delta G_{xx}+\sigma_1G_{yy}+\mathcal{R}_3=0\\ G_t-\sigma_3G_x-\delta F_{xx}-\sigma_1F_{yy}+\mathcal{R}_4=0\\ \mathcal{D}_t-\dfrac{1}{M}\nabla.V-\sigma_1\delta\left((G+G_r)\partial_xH_1-(F+F_r)\partial_xL_1\right)\\ \qquad -\sigma_1^2\left((G+G_r)\partial_yH_2-(F+F_r)\partial_yL_2\right)+\mathcal{R}_5=0\\ \partial_tV-\dfrac{1}{M}\nabla\mathcal{D}+\mathcal{R}_6=0. \end{cases} \tag{4.4}$$

Where

$$\begin{aligned} &\mathcal{R}_1\\ &=-G\nabla\mathcal{D}_r-\tilde{\mathcal{D}}L-\mathcal{D}L_r\\ &\quad-\left(\sigma_2-\frac{\sigma_1}{2}\right)\Bigg(\left(\tilde{F}^2+\tilde{G}^2\right)L+\left(F^2+G^2+2FF_r+2GG_r\right)L_r\\ &\qquad+2G\left(\tilde{H}\tilde{F}+\tilde{G}\tilde{L}\right)+2G_r\left(\tilde{H}F+HF_r+\tilde{L}G+LG_r\right)\Bigg), \end{aligned}$$

$$\begin{aligned} &\mathcal{R}_2\\ &=F\nabla\mathcal{D}_r+\tilde{\mathcal{D}}H+\mathcal{D}H_r\\ &\quad+\left(\sigma_2-\frac{\sigma_1}{2}\right)\Bigg((\tilde{F}^2+\tilde{G}^2)H+\left(F^2+G^2+2FF_r+2GG_r\right)H_r\\ &\qquad+2F\left(\tilde{H}\tilde{F}+\tilde{G}\tilde{L}\right)+2F_r\left(\tilde{H}F+HF_r+\tilde{G}L+GL_r\right)\Bigg), \end{aligned}$$

$$\mathcal{R}_3 = -\Bigg(\left(\sigma_2 - \frac{\sigma_1}{2}\right) \left(\tilde{F}^2 + \tilde{G}^2\right) + \mathcal{D} + \mathcal{D}_r \Bigg) G - \Bigg(\left(\sigma_2 - \frac{\sigma_1}{2}\right) \left(F^2 + G^2 + 2FF_r + 2GG_r\right) + \mathcal{D} \Bigg) G_r,$$

$$\mathcal{R}_4 = \Bigg(\left(\sigma_2 - \frac{\sigma_1}{2}\right) \left(\tilde{F}^2 + \tilde{G}^2\right) + \mathcal{D} + \mathcal{D}_r \Bigg) F + \Bigg(\left(\sigma_2 - \frac{\sigma_1}{2}\right) \left(F^2 + G^2 + 2FF_r + 2GG_r\right) + \mathcal{D} \Bigg) F_r,$$

$$\mathcal{R}_5 = \Bigg(\left(2c_1\tilde{H}_1 + \sigma_1\delta Im(\phi_1)_{xx}\right) F + \left(2c_1\tilde{L} - \sigma_1\delta Re(\phi_1)_{xx}\right) G \Bigg) + 2c_1(H_1F_r + L_1G_r),$$

$$\mathcal{R}_6 = -2\Bigg(c_2(\tilde{H}_1F + F_rH_1 + \tilde{L}_1G + G_rL_1); c_3(\tilde{H}_2F + \tilde{L}_2G + F_rH_2 + G_rL_2) \Bigg)^T.$$

Similarly to the last Section, if $\rho_1\delta > 0$, we can change variables as follows

$$H^* = (\sqrt{\delta\sigma_1}H_1, \sigma_1 H_2) \text{ and } L^* = (\sqrt{\delta\sigma_1}L_1, \sigma_1 L_2),$$

and then we set $U = (H^*, L^*, F, G, \mathcal{D}, V)$. Therefore, the perturbation of (1.2) by the line solitary wave Q is rewritten as a dispersive perturbation of a symmetric hyperbolic given by

$$\begin{aligned} &U_t + (A_1(U) + B_1(\phi_1) + C_1)\, U_x + (A_2(U) + B_2(\phi_1) + C_2)U_y \\ &\quad + (C(U) + \tilde{C}(Q))U = -K_1U_{xx} - K_2U_{yy}. \end{aligned} \tag{4.5}$$

Where, with $j \in \{1,2\}$, A_j, B_j, C_j are symmetric matrices, K_j are skew symmetric and C_j are constant matrices. A_j have the same form as in the proof of Theorem 3.1, $C(U)$ contains quadratic and linear elements, and B_j have the form

$$B_1(\phi_1) = \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & N_1(\phi_1)^T & 0_{3\times2} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times1} & 0_{3\times2} \\ N_1(\phi_1) & 0_{1\times3} & 0 & 0_{1\times2} \\ 0_{2\times3} & 0_{2\times3} & 0_{2\times1} & 0_{2\times2} \end{pmatrix},$$

with

$$N_1(\phi_1) = (-\sqrt{\delta\sigma_1}G_r, 0, \sqrt{\delta\sigma_1}F_r).$$

$$B_2(\phi_1) = \begin{pmatrix} 0 & 0_{1\times3} & 0_{1\times2} & 0 & 0_{1\times2} \\ 0_{3\times1} & 0_{3\times3} & 0_{3\times2} & N_2(\phi_1)^T & 0_{3\times2} \\ 0_{2\times1} & 0_{2\times3} & 0_{2\times2} & 0_{2\times1} & 0_{2\times2} \\ 0 & N_2(\phi_1) & 0_{1\times2} & 0 & 0_{1\times2} \\ 0_{2\times1} & 0_{2\times3} & 0_{2\times2} & 0_{2\times1} & 0_{2\times2} \end{pmatrix}$$

with

$$N_2(\phi_1) = (-\sigma_1G_r, 0, \sigma_1F_r).$$

Furthermore, note that the matrix $\tilde{C}(Q)$ depends on $(\phi_1, \phi_2, \partial_x\phi_3)$ making the following analysis holds for both cases when Q is the bright or the dark soliton.

We have written the perturbation of (1.2) by the line solitary wave Q to the form of a symmetric hyperbolic system. We will apply the same method as in the proof of Theorem 3.1 to obtain the following result.

Theorem 4.1. *Let $\delta\sigma_1 > 0$, $s > 2$ and $s \in \mathbb{N}$. Let*

$$\psi(\cdot,0) = \psi_0 \in H^{s+1}(\mathbb{R}^2); \quad \rho(\cdot,0) = \rho_0 \in H^s(\mathbb{R}^2); \quad \phi(\cdot,0) = \phi_0 \in H^{s+1}(\mathbb{R}^2),$$
$$WM^2(\rho_t + D\phi_{xt} + 2D|\psi|_x^2 + 4DRe(\phi_1\bar{\psi})_x)(\cdot,0) = \nabla.V_0,\ V_0 \in H^s(\mathbb{R}^2).$$

Then there exists $T > 0$ and a unique solution $(\psi,\rho,\phi) \in C([0,T]; H^{s+1}(\mathbb{R}^2)) \times C([0,T]; H^s(\mathbb{R}^2)) \times C([0,T]; H^{s+1}(\mathbb{R}^2))$ of the (1.2) when it is perturbed by the line soliton $Q = (\phi_1,\phi_2,\phi_3)$.

Proof. The proof of Theorem 4.1 and Theorem 3.1 are essentially the same except the estimates for the terms $B_j(\phi_1)$ and $\tilde{C}(Q)$.
With the same notations as in the proof of Theorem 3.1 we consider the following iteration scheme

$$\begin{cases} U^0(x,y,t) = U_0^0(x,y) \\ U_t^{k+1} + (A_1(U^k) + B_1(\phi_1) + C_1)U_x^{k+1} + (A_2(U^k) + B_2(\phi_1) + C_2)U_y^{k+1} \\ \qquad + (C(U^k) + \tilde{C}(Q))U^{k+1} = -K_1U_{xx}^{k+1} - K_2U_{yy}^{k+1} \\ U^{k+1}(x,y,0) = U_0^{k+1}(x,y). \end{cases} \tag{4.6}$$

Denoting $V = \nabla^s U$, then applying ∇^s to the second equation of (4.6) we get

$$\begin{aligned} &V_t^{k+1} + A_1(U^k)V_x^{k+1} + [\nabla^s, A_1(U^k)]U_x^{k+1} + \nabla^s\left((B_1(\phi_1) + C_1)U_x^{k+1}\right) \\ &\quad + A_2(U^k)V_y^{k+1} + [\nabla^s, A_2(U^k)]U_y^{k+1} + \nabla^s\left((B_2(\phi_1) + C_2)U_y^{k+1}\right) \\ &\quad + C(U^k)V^{k+1} + [\nabla^s, C(U^k)]U^{k+1} + \nabla^s\left(\tilde{C}(Q)U^{k+1}\right) \\ &= -K_1V_{xx}^{k+1} - K_2V_{yy}^{k+1}. \end{aligned} \tag{4.7}$$

We multiply (4.7) with V^{k+1} and apply the similar argument as for (3.10) with these following modifications:

a) We estimate the terms which do not depend on ϕ_1 by using the following commutative estimate instead of (3.11).

Lemma 4.1. *(see [24].B 2) Let $s > d/2 + 1$. If $f \in H^s(\mathbb{R}^d)$ then, for all $g \in H^{s-1}$ then*

$$\|[\nabla^s(D), f], g\|_{L^2} \le C\left(\|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}}\|g\|_{L^\infty}\right). \tag{4.8}$$

b) We can not use (3.11) with the differential operator $J^s = (1-\Delta)^{s/2}$, because ϕ_1 is one variable function and does not decay at infinity. In order to overcome this difficulty we therefore consider ∇^s with integer order. Combining the Leibniz rule, Hölder in equality, integral by part and the fact that B_1, B_2 are symmetric we get

$$\begin{aligned} &\sum_{j=1,2}\left\langle \nabla^s\left(B_j(\phi_1)U_x^{k+1}\right), V^{k+1}\right\rangle + \left\langle \nabla^s\left(\tilde{C}(Q)U^{k+1}\right), V^{k+1}\right\rangle \\ &\quad \le C\left(\|\phi_1\|_{W^{s,\infty}}, \|\phi_2\|_{W^{s,\infty}}, \|\partial_x\phi_3\|_{W^{s,\infty}}\right)\left\|U^{k+1}\right\|_{H^s}^2. \end{aligned}$$

c) Note that we can get rid the term involving C_1 and C_2 since they are constant and symmetric matrices.

The rest of the proof proceeds exactly as in the proof of Theorem 3.1. $\square$

5. Global solution

In this section we will establish the conservation of energy for the perturbation of (1.2) by the line soliton Q given in Section 2 and as a consequence, the existence of a global weak solution when Q is the dark soliton.

In order to make the calculation easier, we will consider the solution of the form

$$(e^{i\lambda t}e^{i\frac{\sigma_3}{2\delta}x}\psi(x,y,t), \rho(x,y,t), \phi(x,y,t))$$

then the 1-d solitary wave will have the following form

$$Q = (\phi_1, \phi_2, \phi_3) = (R(x), aR^2(x), b\int R^2(x)),$$

with $R(x)$ given in (2.6) or (2.8) (note that this trick will not affect the analysis in section 4).

Then the perturbed system is

$$\begin{cases} \psi_t + i(\lambda - \frac{\sigma_3^2}{4\delta})\psi - i\delta\psi_{xx} - i\sigma_1\psi_{yy} \\ \quad + i\left\{\sigma_2|\psi|^2 + W(\rho + D\rho_x) + 2\sigma_2\phi_1 Re(\psi) + |\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)\right\}\psi \\ \quad + i\{\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1 Re(\psi)\}\phi_1 = 0 \\ \rho_t + \Delta\phi + D(|\psi|^2 + 2\phi_1 Re(\psi))_x = 0 \\ \phi_t + \frac{1}{M^2}\rho + |\psi|^2 + 2\phi_1 Re(\psi) = 0, \end{cases} \tag{5.1}$$

In this section we will establish the energy conservation for (5.1) when it is perturbed by a line soliton Q and the existence of a global weak solution when Q is the dark soliton.

Theorem 5.1. *Let (ψ, ρ, ϕ) be a solution of system (5.1) obtained in Theorem 4.1, defined in the time interval $[0,T]$. Then the quantity*

$$\begin{aligned} E = &(\lambda - \frac{\sigma_3^2}{4\delta})\|\psi\|_{L^2}^2 + \delta\|\psi_x\|_{L^2}^2 + \sigma_1\|\psi_y\|_{L^2}^2 \\ &+ \frac{\sigma_2}{2}\left\||\psi|^2 + 2\phi_1 Re(\psi)\right\|_{L^2}^2 + \sigma_2\|\phi_1\psi\|_{L^2}^2 + \frac{W}{2M^2}\|\rho\|_{L^2}^2 + \frac{W}{2}\|\nabla\phi\|_{L^2}^2 \\ &- \int W(M^2 + D^2)|\phi_1|^2|\psi|^2 + \int W(\rho + D\phi_x)(|\psi|^2 + 2\phi_1 Re(\psi)). \end{aligned} \tag{5.2}$$

is conserved for $t \in [0,T]$.

Proof. We multiply the first equation in (5.1) by $\partial_t\bar{\psi}$, integrate the result and take its imaginary part to get successively

$$(\lambda - \frac{\sigma_3^2}{4\delta})Re\int \psi\bar{\psi}_t = \frac{1}{2}(\lambda - \frac{\sigma_3^2}{4\delta})\int |\psi|_t^2, \tag{5.3}$$

$$-\delta Re\int \psi_{xx}\bar{\psi}_t = \delta Re\int \psi_x(\bar{\psi}_x)_t = \frac{1}{2}\delta\int |\psi_x|_t^2, \tag{5.4}$$

$$-\sigma_1 Re\int \psi_{yy}\bar{\psi}_t = \frac{1}{2}\sigma_1\int |\psi_y|_t^2, \tag{5.5}$$

(5.6)
$$\begin{aligned} &Re\int \left(\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1 Re(\psi) + \sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)\right)\psi\bar{\psi}_t \\ &\quad = \frac{1}{2}\int \left(\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1 Re(\psi) + \sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)\right)|\psi|_t^2 \\ &\quad = \frac{1}{2}\int \frac{1}{2}\sigma_2|\psi|_t^4 + W(\rho + D\phi_x)|\psi|_t^2 + 2\sigma_2\phi_1 Re(\psi)|\psi|_t^2 \\ &\qquad + \left(\left(\sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)\right)|\psi|^2\right)_t, \end{aligned}$$

$$\begin{aligned} &Re\int \left(\sigma_2|\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2\phi_1 Re(\psi)\right)\phi_1\bar{\psi}_t \\ &\quad = \int \sigma_2|\psi|^2(\phi_1 Re(\psi))_t + W(\rho + D\phi_x)(\phi_1 Re(\psi))_t + \sigma_2|\phi_1 Re(\psi)|_t^2. \end{aligned} \tag{5.7}$$

Combining (5.3), (5.4), (5.5), (5.6) and (5.7) we obtain

$$\begin{aligned} 0 = \frac{d}{dt}\int \Bigg(&\frac{1}{2}(\lambda - \frac{\sigma_3^2}{4\delta})|\psi|^2 + \frac{\delta}{2}|\psi_x|^2 + \frac{\sigma_1}{2}|\psi_y|^2 + \frac{\sigma_2}{4}|\psi|^4 \\ &+ \sigma_2|\psi|^2\phi_1 Re(\psi) + \frac{1}{2}\left(\sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)\right)|\psi|^2 + \sigma_2|\phi_1 Re(\psi)|^2 \\ &+ \frac{1}{2}W(\rho + D\phi_x)(|\psi|^2 + 2\phi_1 Re(\psi))\Bigg) \\ &- \frac{1}{2}\int W(\rho_t + D\partial_t\phi_x)(|\psi|^2 + 2\phi_1 Re(\psi)) \end{aligned}$$

From the second and the third equation in (5.1), we get

$$\int \rho_t(|\psi|^2 + 2\phi_1 Re(\psi)) = -\int \rho_t(\phi_t + \frac{1}{M^2}\rho) = -\int \rho_t\phi_t - \frac{1}{2M^2}\int \rho_t^2,$$

and

$$\begin{aligned} \int D\partial_t\phi_x(|\psi|^2 + 2\phi_1 Re(\psi)) &= -\int \phi_t\, D(|\psi|^2 + 2\phi_1 Re(\psi))_x \\ &= \int \phi_t(\rho_t + \Delta\phi) \\ &= \int \phi_t\rho_t - \frac{1}{2}\int |\nabla\phi|_t^2. \end{aligned}$$

That implies

$$
\begin{aligned}
0 = \frac{d}{dt}\int \Big(& \frac{1}{2}(\lambda - \frac{\sigma_3^2}{4\delta})|\psi|^2 + \frac{\delta}{2}|\psi_x|^2 + \frac{\sigma_1}{2}|\psi_y|^2 + \frac{\rho_2}{4}|\psi|^4 \\
& + \sigma_2|\psi|^2\phi_1 Re(\psi) + \frac{1}{2}\left(\sigma_2|\phi_1|^2 + W(\phi_2 + D\partial_x\phi_3)\right)|\psi|^2 + \sigma_2|\phi_1 Re(\psi)|^2 \\
& + \frac{1}{2}W(\rho + D\phi_x)(|\psi|^2 + 2\phi_1 Re(\psi)) \\
& + \frac{W}{4M^2}\rho^2 + \frac{W}{4}|\nabla\phi|^2 \Big)
\end{aligned}
$$

Finally we get the energy conservation

$$
\begin{aligned}
E = & (\lambda - \frac{\sigma_3^2}{4\delta})\,\|\psi\|_{L^2}^2 + \delta\,\|\psi_x\|_{L^2}^2 + \sigma_1\,\|\psi_y\|_{L^2}^2 + \frac{\sigma_2}{2}\,\|\psi\|_{L^4}^4 \\
& + 2\sigma_2\int |\psi|^2\phi_1 Re(\psi) + \sigma_2\,\|\phi_1\psi\|_{L^2}^2 + 2\sigma_2\,\|\phi_1 Re(\psi)\|_{L^2}^2 \\
& + \int W(\phi_2 + D\partial_x\phi_3)|\psi|^2 + \frac{W}{2M^2}\,\|\rho\|_{L^2}^2 + \frac{W}{2}\,\|\nabla\phi\|_{L^2}^2 \\
& + \int W(\rho + D\phi_x)(|\psi|^2 + 2\phi_1 Re(\psi)) \\
= & (\lambda - \frac{\sigma_3^2}{4\delta})\,\|\psi\|_{L^2}^2 + \delta\,\|\psi_x\|_{L^2}^2 + \sigma_1\,\|\psi_y\|_{L^2}^2 \\
& + \frac{\sigma_2}{2}\,\big\||\psi|^2 + 2\phi_1 Re(\psi)\big\|_{L^2}^2 + \sigma_2\,\|\phi_1\psi\|_{L^2}^2 + \frac{W}{2M^2}\,\|\rho\|_{L^2}^2 + \frac{W}{2}\,\|\nabla\phi\|_{L^2}^2 \\
& - \int W(M^2 + D^2)|\phi_1|^2|\psi|^2 + \int W(\rho + D\phi_x)(|\psi|^2 + 2\phi_1 Re(\psi)).
\end{aligned}
$$

Note that we got rid of the terms involving ϕ_2 and ϕ_3 since by (2.3) one has $a + Db = -(M^2 + D^2)$ implying that $\phi_2 + D\partial_x\phi_3 = -(M^2 + D^2)\phi_1^2$. □

Theorem 5.2. *Assume that Q is the dark soliton given by (2.8) with wave speed $c = 0$.*
i) Let (ψ, ρ, ϕ) be the solution of (5.1) obtained by Theorem 4.1 with existence time interval $[0, T]$. Then for all $t \in [0, T]$ we have

$$
\|\psi(t)\|_{H^1} + \|\rho(t)\|_{L^2} + \|\phi(t)\|_{H^1} \leq C(t). \tag{5.8}
$$

ii) For any $(\psi_0, \rho_0, \phi_0) \in H^1 \times L^2 \times H^1$, there exists a global weak solution (ψ, ρ, ϕ) of (5.1) such that for any $T > 0$

$$
\begin{aligned}
& \psi, \phi \in L^\infty([0,T] : H^1),\ \rho \in L^\infty([0,T] : L^2) \\
& \psi_t, \rho_t \in L^\infty([0,T] : H^{-1}),\ \phi_t \in L^\infty([0,T] : L^2).
\end{aligned} \tag{5.9}
$$

Proof. i) For any $\varepsilon \in (0, 1)$, using Cauchy inequality we have

$$
\left|\int W\rho(|\psi|^2 + 2\phi_1 Re(\psi))\right| \leq \frac{W}{2M^2}(1-\varepsilon)\int \rho^2 + \frac{M^2 W}{2(1-\varepsilon)}\int (|\psi|^2 + 2\phi_1 Re(\psi))^2
$$

and

$$
\left|\int WD\phi_x(|\psi|^2 + 2\phi_1 Re(\psi))\right| \leq \frac{W}{2}(1-\varepsilon)\int |\phi_x|^2 + \frac{WD^2}{2(1-\varepsilon)}\int (|\psi|^2 + 2\phi_1 Re(\psi))^2,
$$

then

$$\left| \int W(\rho + D\phi_x)(|\psi|^2 + 2\phi_1 Re(\psi)) \right|$$
$$\leq \frac{W(M^2+D^2)}{2(1-\varepsilon)} \left\| |\psi|^2 + 2\phi_1 Re(\psi) \right\|_{L^2}^2 + \frac{W}{2M^2}(1-\varepsilon) \|\rho\|_{L^2}^2$$
$$+ \frac{W}{2}(1-\varepsilon) \|\nabla\phi\|_{L^2}^2 .$$

Note that we are considering the stationary dark soliton, so that the condition (2.7) becomes

$$\begin{cases} \frac{1}{\delta}\left(\frac{\sigma_3^2}{4\delta} - \lambda\right) > 0 \\ \frac{1}{\delta}\left(\sigma_2 - W(M^2+D^2)\right) > 0, \end{cases}$$

so there exists $\varepsilon > 0$ small enough such that

$$\sigma_2 > \frac{W(M^2+D^2)}{(1-\varepsilon)}.$$

Therefore, the conservation law (5.2) implies

(5.10)
$$\frac{1}{2}(\sigma_2 - \frac{W(M^2+D^2)}{(1-\varepsilon)}) \left\| |\psi|^2 + 2\phi_1 Re(\psi) \right\|_{L^2}^2 + \varepsilon \frac{W}{2M^2} \|\rho\|_{L^2}^2 + \varepsilon \frac{W}{2} \|\nabla\phi\|_{L^2}^2$$
$$\leq E + (\frac{\sigma_3^2}{4\delta} - \lambda) \|\psi\|_{L^2}^2 .$$

From now we will fix such an ε and define

$$c_1 = \frac{1}{2}(\sigma_2 - \frac{W(M+D^2)}{(1-\varepsilon)}),\ c_2 = \varepsilon\frac{W}{2M},\ c_3 = \varepsilon\frac{W}{2}.$$

The first equation of (5.1) implies

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 = Im \int (\sigma_2(|\psi|^2 + 2\phi_1 Re(\psi)) + W(\rho + D\phi_x))\phi_1\bar{\psi}$$
$$\leq c_1 \left\| |\psi|^2 + 2\phi_1 Re(\psi) \right\|_{L^2}^2 + c_2 \|\rho\|_{L^2}^2 + c_3 \|\phi_x\|_{L^2}^2$$
$$+ (\frac{\sigma_2^2}{4c_1} + \frac{W^2}{4c_2} + \frac{W^2D^2}{4c_3}) \|\phi_1 Im(\psi)\|_{L^2}^2 .$$

Combining with (5.10) we get

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 \leq E + (\frac{\sigma_3^2}{4\delta} - \lambda)\|\psi\|_{L^2}^2$$
$$+ (\frac{\sigma_2^2}{4c_1} + \frac{W^2}{4c_2} + \frac{W^2D^2}{4c_3}) \|\phi_1 Im(\psi)\|_{L^2}^2 .$$

Since $\phi_1 \in L^\infty$, $\|\phi_1 Im(\psi)\|_{L^2}$ is under control and we recall that $\frac{\sigma_3^2}{4\delta} - \lambda > 0$. Hence, by Gronwall inequality we can bound $\|\psi(t)\|_{L^2}$ by a constant depending on t. The bounds on $\|\nabla\psi\|_{L^2}$, $\|\rho\|_{L^2}$, $\|\nabla\phi\|_{L^2}$ then follow from the energy conservation.

ii) We shall use a classical compactness method. First we approximate (ψ_0, ρ_0, ϕ_0) by smooth functions $(\psi_{o,\eta}, \rho_{0,\eta}, \phi_{0,\eta})$ and obtain local approximate solutions

$(\psi_\eta, \rho_\eta, \phi_\eta)$ on some time interval $[0, T_\eta]$ due to Theorem 4.1.
Using part i), we have that for any $T > 0$, $(\psi_\eta, \rho_\eta, \phi_\eta)$ is bounded, independently of η, in the space

$$L^\infty((0,T) : H^1) \times L^\infty((0,T) : L^2) \times L^\infty((0,T) : H^1). \tag{5.11}$$

Therefore, using (5.1) and Sobolev theorem we infer that $(\partial_t\psi_\eta, \partial_t\rho_\eta, \partial_t\phi_\eta)$ is bounded in

$$L^\infty((0,T) : H^{-1}) \times L^\infty((0,T) : H^{-1}) \times L^\infty((0,T) : L^2).$$

Hence, up to a subsequence one can assume that

$$\begin{aligned} &\psi_\eta \to \psi \text{ in } L^\infty((0,T) : H^1) - \text{weak*}, \\ &\rho_\eta \to \rho \text{ in } L^\infty((0,T) : L^2) - \text{weak*}, \\ &\phi_\eta \to \phi \text{ in } L^\infty((0,T) : H^1) - \text{weak*}. \end{aligned} \tag{5.12}$$

By Aubin-Lions lemma one can furthermore assume that up to a subsequence

$$\psi_\eta \to \psi \text{ in } L^p_{loc}([0,T] : L^q_{loc}(\mathbb{R}^2)), \tag{5.13}$$

for any $2 \leq p, q < \infty$. Similar convergence results hold true for ϕ_η, ρ_η.

These convergences allow to pass to the limit in the distribution sense in (5.1) for $(\psi_\eta, \rho_\eta, \phi_\eta)$, proving that (ψ, ρ, ϕ) satisfies (5.1) in $L^\infty((0,T) : H^{-1}) \times L^\infty((0,T) : H^{-1}) \times L^\infty((0,T) : L^2)$.
The initial condition makes sense since

$$(\psi, \rho, \phi) \in C_w([0,T] : H^1) \times C_w([0,T] : L^2) \times C_w([0,T] : H^1)$$

□

6. Solitary wave solutions in higher dimension

Let consider now solitary waves solutions of (1.2) that is solutions of the form $(e^{i\omega t}\psi(x + \sigma_3 t, y), \phi(x,y), \eta(x,y)),\ \omega \in \mathbb{R},\ \psi \in H^1(\mathbb{R}^2)$, yielding the system

$$\begin{cases} -\omega\psi + \delta\psi_{xx} + \sigma_1\nabla_\perp\psi - (\sigma_2 - WM^2)|\psi|^2\psi - cW\phi_x\psi = 0 \\ \Delta\phi + c|\psi|^2_x = 0, \end{cases} \tag{6.1}$$

which is similar to the equation for the solitary wave solutions of the elliptic/hyperbolic-elliptic Davey -Stewartson systems in the terminology of [16]. By Pohojaev type arguments one obtains (see [17] for similar arguments) that non trivial solutions to (6.1) cannot exist when $\delta\sigma_1 < 0$.

On the other hand the existence of non trivial solutions to (6.1) has been established ([9]) in the *focusing case*

$$\delta\sigma_1 > 0, \quad c < 0, \quad c(W(\sigma_2 - WM^2) < 0.$$

Various stability and instability results of solutions to (6.1) have been obtained in [10, 30, 31, 32] in the context of the Davey-Stewartson systems but no similar results seemed to be known when they are viewed as solutions to the Zakharov-Rubenchik systems. In particular one does not know if the solutions of (6.1) are constrainded minimizers of the Zakharov-Rubenchik system.

According to the Davey-Stewartson case, one could conjecture that those localized solitary waves are unstable.

7. Conclusion and open questions

We have addressed in this paper some issues on the Zakharov-Rubenchik, Benney-Roskes systems. Many questions remain unsolved for those important systems and we indicate a few below.

1. Justify rigorously the limit of ZR (BR) systems to the Davey-Stewartson systems. This is non trivial (because of a boundary layer at $t = 0$) issue is analogous (but more delicate) to the Schrödinger limit of the Zakharov system (see [36, 37]).

2. The present work can be viewed as a preliminary step towards the study of the transverse stability/instability of the ZR or BR one dimensional dark or bright solitary wave. Perturbations could be localized in (x,y) or periodic in y. The Cauchy problem was addressed in the present paper in both cases.

In both the functional settings, we plan to come back to those transverse stability issues a subsequent work, in the spirit of [40, 41, 42].

3. It is known ([19, 20]) that (radially symmetric) solutions of the Zakharov system may blow up in finite time. Such a result is unknown for the Zakharov-Rubenchik, Benney-Roskes system and it would interesting to see if the results in [19, 20] extend to (1.1).

4. The existence result Theorem 3.1 is established when $\delta\sigma_1 > 0$, that is when the Schrödinger equation in (1.1) is not an "non elliptic" one in the terminology of [16]. This condition is never satisfied in the context of purely gravity waves water waves (see [25] and the discussion above) and it would interesting to relax it.

5. We recall that an existence result on time scales of order $1/\epsilon$ is needed to fullly justify the Benney-Roskes system. Obtaining such a result is still a challenging open problem.

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