

LOCAL WELL-POSEDNESS FOR THE ZAKHAROV SYSTEM ON THE BACKGROUND OF A LINE SOLITON

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ABSTRACT. We prove that the Cauchy problem for the two-dimensional Zakharov system is locally well-posed for initial data which are localized perturbations of a line solitary wave. Furthermore, for this Zakharov system, we prove weak convergence to a nonlinear Schrödinger equation.

1. INTRODUCTION

In this paper, we study the initial value problem for the scalar version of the two dimensional Zakharov system

$$i\partial_t u + \Delta u = nu \tag{1.1}$$

$$\frac{1}{\lambda^2} \partial_t^2 n - \Delta n = \Delta |u|^2, \tag{1.2}$$

where $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$, λ is a fixed number, u is a complex valued function, n is a real function, with initial data

$$u(x, y, 0) = u_0(x, y) + Q(x), \quad n(x, y, 0) = n_0(x, y) - Q(x)^2, \quad n_t(x, y, 0) = n_1(x, y). \tag{1.3}$$

The above function $Q(x) = 2\sqrt{2}/(e^x + e^{-x})$ is the unique positive solution of the equation

$$Q_{xx} - Q + Q^3 = 0,$$

and

$$(e^{it}Q(x), -Q(x)^2) \tag{1.4}$$

is a line soliton of (1.1)-(1.2).

And also note that $e^{it}Q(x)$ is a line soliton of the cubic focusing nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u + |u|^2 u = 0. \tag{1.5}$$

The Zakharov system is introduced in [22] to describe the propagation of Langmuir waves in plasma. For more details about the derivation and the physical background of Zakharov system we refer to [20] and [22].

In (1.4) the soliton is considered as a two dimensional (constant in y) object. A natural question is that of its transverse (with respect to y) stability. When looking for *localized* perturbations, one is lead, as a first step, to study the Cauchy problem for the perturbed system below.

Another possibility, looking for *y-periodic* perturbations, would be to study (1.1)-(1.2) as a system posed on $\mathbb{R} \times \mathbb{T}$, the 1-d torus case was studied in [4, 21]. On the other hand, the Zakharov system in spatial space \mathbb{R}^2 and \mathbb{R}^3 have been studied by many authors [1, 2, 3, 5, 6, 7, 8, 9, 18, 19]

The local well-posedness of (1.1)-(1.2) with initial data given by (1.3) which will be proved in this paper can be viewed as the first step to study the transverse

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stability (or instability) of the line soliton (1.4). As far as we know, this problem is still an open problem. Concerning *global* issues we can quote [7, 8] where the authors prove the existence of self-similar blow-up solutions and the instability by blow-up of periodic (in time) solutions (solitary wave solutions) of the Zakharov system with data in $H^1(\mathbb{R}^2)$. Their method uses the radial symmetry of the system which is broken when one writes the system satisfied by a localized perturbation of the line solitary wave.

The transverse instability of the line solitary wave for some two dimensional models such as nonlinear Schrödinger equation, Kadomtsev-Petviashvili equation and for some general “abstract” problems have been studied extensively in [15, 16, 17] but the framework of those papers does not seem to include the case of the Zakharov system.

In other way, instead of the system (1.1)-(1.2) with the initial data of the form (1.3) we can consider the following perturbed system

$$i\partial_t(u + Q) - (u + Q) + \Delta(u + Q) = (n - Q^2)(u + Q) \quad (1.6)$$

$$\frac{1}{\lambda^2}\partial_t^2(n - Q^2) - \Delta(n - Q^2) = \Delta|u + Q|^2, \quad (1.7)$$

with initial data

$$u(x, y, 0) = u_0, \quad n(x, y, 0) = n_0, \quad n_t(x, y, 0) = n_1. \quad (1.8)$$

Note that in the above system, we used the change of variable $\bar{u}(x, y, t) := e^{it}u(x, y, t)$.

The main purpose of this paper is to prove that the Cauchy problem (1.6), (1.7), (1.8) is locally well-posed in a suitable functional framework. The main differences between the Zakharov system and its perturbation lie in the new terms containing “ Q ”. More precisely,

i) If we reduce system (1.6)-(1.7) to a nonlinear Schrödinger equation with loss of derivative in the nonlinearity, then we can think of using the smoothing effect of Schrödinger operator as in [9]. In our case, the linear terms Qn and Q^2u will give trouble because the function Q does not decay in y at infinity.

ii) We use the method of Bourgain (actually the techniques developed in [5, 6]) and the method of Schochet-Weinstein (see [18]) to obtain two versions of the local well-posedness result for system (1.6)-(1.7). The differences with the case of the unperturbed Zakharov system are the estimates for the new linear terms and the effects of Q in each method.

Compared to Bourgain method, Schochet-Weinstein method provides well-posedness in smaller Sobolev spaces but allows, when a suitable small parameter is included, to obtain the Schrödinger limit. For the unperturbed system see [2, 14, 18].

iii) There are some difficulties concerning blow-up and global existence issues. In the setting of (1.6)-(1.7), the approach in [7], [8] does not make sense because we do not have radial symmetric solutions. We also do not have L^2 conservation or at least a bound on L^2 norm that makes the usual method to extend the local solution impossible to apply.

The rest of this paper is organized as follows.

In Section 2, we present the energy conservation of system (1.6)-(1.7).

In Section 3, we assume that $\lambda = 1$ and use method to prove the local well-posedness of system (1.6)-(1.7), more precisely we have the following theorem.

Theorem 1. *Let k and l satisfy*

$$k > l \geq 0, \quad l + 1 \geq k \geq \frac{l + 1}{2}. \quad (1.9)$$

Then system (1.6)-(1.7) with initial data $(u_0, n_0, n_1) \in H^k \times H^l \times H^{l-1}$ is locally well posed in $X_1^{k, b_1} \times X_2^{l, b} \times X_2^{l-1, b}$ for suitable b, b_1 close to $1/2$.

Furthermore the solutions satisfy

$$(u, n, \partial_t n) \in \mathcal{C}([0, T], H^k \times H^l \times H^{l-1}), \quad (1.10)$$

where T is the existence time.

(The Bourgain space $X_j^{s,b}$ will be defined as (3.8) in section 3.)

In Section 4, we present the proof by using the method of Schochet and Weinstein. By using this method, we can get a local (in time) bound in a suitable Sobolev space that allows us to pass to limit to a nonlinear Schrödinger type equation (NLS) when λ tends to ∞ . Comparing to the results in [18] for Zakharov system (1.1)-(1.2), the result for the perturbed system is weaker, because of the appearance of the linear terms containing “ Q ”. We will see that difference in the proof of Theorem 3. In particular we have the following theorem.

Theorem 2. *Let $s > 2$, then the initial value problem (1.6)-(1.7) with initial data of the form*

$$u(0) = u_0, \quad n(0) = n_0, \quad n_t(0) = \nabla \cdot f_0,$$

has a unique solution $(u, n, n_t) \in L^\infty([0, T]; H^{s+1}(\mathbb{R}^2)) \times L^\infty([0, T]; H^s(\mathbb{R}^2)) \times L^\infty([0, T]; H^s(\mathbb{R}^2))$, where $[0, T]$ is the time interval of existence.

Furthermore, suppose that

$$\|u_0(\lambda)\|_{H^{s+1}} + \|n_0(\lambda)\|_{H^s} + \frac{1}{\lambda} \|f_0(\lambda)\|_{H^s} \leq C_1. \quad (1.11)$$

Then T depends only on C_1 . Moreover, the solution (u, n) satisfies

$$\begin{aligned} \|u(t, \lambda)\|_{H^{s+1}} + \|u_t(t, \lambda)\|_{H^{s-1}} + \|n(t, \lambda)\|_{H^s} + \frac{1}{\lambda} \|n_t(t, \lambda)\|_{H^{s-1}} \\ + \frac{1}{\lambda^2} \|n_{tt}(t, \lambda)\|_{H^{s-2}} \leq C_2, \end{aligned} \quad (1.12)$$

for all $t \in [0, T]$.

In Section 5, we establish a weak convergence result stating that the local solution $(u(\lambda), n(\lambda))$ of (1.6)-(1.7) with initial data (1.8) tends to $(u, -|u|^2)$ weakly when λ tends to ∞ . Where u is the unique solution of the following perturbation of the nonlinear Schrödinger equation

$$i(u + Q)_t - (u + Q) + \Delta(u + Q) + |u + Q|^2(u + Q) = 0 \quad (1.13)$$

with $u(x, y, 0) = u_0$.

Remark: The local well-posedness of (1.13) in a Sobolev space H^s with $s \geq 1$ can be obtained by using Strichartz estimate. Unfortunately, the global existence in the focusing case is still unknown to us.

Notations:

\mathcal{F} , \mathcal{F}_t , \mathcal{F}_x , \mathcal{F}_y and \mathcal{F}^{-1} : the Fourier transform of a function in spacetime, time, space and the inverse Fourier transform respectively. The hat “ $\hat{\cdot}$ ” is also used as the short notation of Fourier transform.

H^s is the usual Sobolev space.

$|(\xi_1, \xi_2)| = (\xi_1^2 + \xi_2^2)^{1/2}$, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$.

$D_x = -i d/dx$.

$\|\cdot\|_2$: The L^2 norm in space and time.

$\|u; X\|$ or $\|u\|_X$: The norm of function u in functional space X .

C will be a general constant unless otherwise explicitly indicated.

2. CONSERVATION LAW

System (1.6)-(1.7) can be rewritten as follows

$$i\partial_t u + \Delta u - u = nu + Qn - Q^2 u \quad (2.1)$$

$$n_t = \lambda^2 \nabla \cdot \mathbf{v} \quad (2.2)$$

$$\partial_t \mathbf{v} - \nabla n = \nabla |u|^2 + 2\nabla(QRe(u)). \quad (2.3)$$

Proposition 2.1. *Let (u, n, \mathbf{v}) be a solution of system (2.1)-(2.3) obtained in Theorem 1, defined in the time interval $[0, T]$. Then*

$$\frac{d}{dt} E(t) = 0, \quad 0 \leq t \leq T, \quad (2.4)$$

where

$$E(t) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |u|^2 + \frac{n^2}{2} + \frac{\lambda^2 |\mathbf{v}|^2}{2} - Q^2 |u|^2 + n(|u|^2 + 2QRe(u)) \right) dx dy \quad (2.5)$$

Proof. To obtain (2.4) we proceed formally. A rigorous proof can be obtained by smoothing the initial data and passing to the appropriate limit.

We multiply (2.1) by $\partial_t \bar{u}$, integrate and take its real part to get

$$\int \partial_t (|\nabla u|^2 + |u|^2 - Q^2 |u|^2) + n \partial_t (|u|^2 + 2QRe(u)) = 0. \quad (2.6)$$

We also multiply (2.3) by $\nabla^{-1} n_t$ or $\lambda^2 \mathbf{v}$ and integrate to get

$$\int \partial_t \left(\frac{\lambda^2 |\mathbf{v}|^2}{2} + \frac{n^2}{2} \right) + (|u|^2 + 2QRe(u)) n_t = 0. \quad (2.7)$$

Combining (2.6) and (2.7) we obtain (2.4). \square

Remark: The energy space of system (1.6)-(1.7) is $H^1 \times L^2 \times H^{-1}$ and Theorem 1 gives the local well-posedness of (1.6)-(1.7) in the energy space. It is expected to get a global solution with small initial data in the energy space, but in our case, the difficulty is the lack of L^2 conservation.

3. BOURGAIN METHOD

3.1. Linear estimate. Throughout this Section we will assume that $\lambda = 1$ and then split the function n in (1.6)-(1.7) into two parts

$$n = n_+ + n_-,$$

where $n_{\pm} = n \pm i\omega^{-1} \partial_t n$, $\omega = (-\Delta)^{1/2}$,

so that the system (1.6)-(1.7) can be rewritten as

$$i\partial_t u = -\Delta u + u + \frac{n_+ + n_-}{2} u + \frac{n_+ + n_-}{2} Q - Q^2 u \quad (3.1)$$

$$(i\partial_t \mp \omega) n_{\pm} = \pm \omega |u|^2 \pm 2\omega(QRe(u)) \quad (3.2)$$

Equations (3.1) and (3.2) have the form

$$i\partial_t u = \phi(-i\nabla)u + f(u), \quad (3.3)$$

where ϕ is a real function defined in \mathbb{R}^2 and f some nonlinear function. The Cauchy problem for (3.3) with initial data u_0 is rewritten as the integral equation

$$\begin{aligned} u(t) &= U(t)u_0 - i \int_0^t dt' U(t-t') f(u(t')) \\ &= U(t)u_0 - iU *_R f(u), \end{aligned} \quad (3.4)$$

where $U(t) = e^{-it\phi(-i\nabla)}$ and $*_R$ denotes the retarded convolution in time.

Let $\psi_1 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ be even with $0 \leq \psi_1 \leq 1$, $\psi_1(t) = 1$ for $|t| < 1$, $\psi_1(t) = 0$ for $|t| \geq 2$ and let $\psi_T = \psi_1(t/T)$ for $0 < T \leq 1$. One then replaces equation (3.4) by the cut off equation

$$u(t) = \psi_1(t)U(t)u_0 - i\psi_T(t) \int_0^t dt' U(t-t')f(u(t')). \quad (3.5)$$

Then we have the cut off integral equations associated with (3.1) and (3.2) namely

$$u(t) = \psi_1(t)U(t)u_0 - (i/2)\psi_T(t) \int_0^t dt' U(t-t')(u + (u+Q)(n_+ + n_-) - 2|Q|^2u)(t') \quad (3.6)$$

$$n_\pm(t) = \psi_1(t)V_\pm(t)n_{\pm 0} \mp i\psi_T(t) \int_0^t dt' V_\pm(t-t')\omega(|u|^2 + 2QRe(u))(t'), \quad (3.7)$$

where $U(t) = e^{it\Delta}$ and $V_\pm(t) = e^{\mp i\omega t}$.

We use a standard contraction method on the two operators on the right hand side of (3.6)-(3.7) with $u \in X_1^{k,b_1}$ and $n_\pm \in X_2^{l,b}$. Where X_1^{k,b_1} and $X_2^{l,b}$ are the Bourgain spaces associated to two operators with symbols $\phi_1(\xi) = |\xi|^2$ and $\phi_2(\xi) = \pm|\xi|$, which are given by the following definition

$$\|u; X_j^{s,b}\| = \left\| \langle \xi \rangle^s \langle \tau + \phi_j(\xi) \rangle^b \widehat{u}(\xi, \tau) \right\|_2, \quad (3.8)$$

where $\phi_j(\xi)$ is the symbol of the associated differential operator.

We will also use the following definition of space time Sobolev space,

$$\|u; H^{s,b}\| = \left\| \langle \xi \rangle^s \langle \tau \rangle^b \widehat{u}(\xi, \tau) \right\|_2.$$

The linear estimate is given by the following lemma

Lemma 3.1. (cf. [6] Lemma 2.1) (i) Let $b' \leq 0 \leq b \leq b' + 1$ and $T \leq 1$. Then

$$\begin{aligned} & \|\psi_T U *_R f; X^{s,b}\| \\ & \leq C\sqrt{2}\{T^{1-b+b'} \|f; X^{s,b'}\| + T^{1/2-b} \|\mathcal{F}^{-1}\chi(|\tau|T \geq 1)\mathcal{F}f; Y^s\| \}, \end{aligned} \quad (3.9)$$

where

$$\|f; Y^s\| = \left\| \langle \xi \rangle^s \langle \tau + \phi(\xi) \rangle^{-1} \widehat{f}(\xi, \tau); L_\xi^2 L_\tau^1 \right\|.$$

(ii) Suppose in addition that $b' > -1/2$. Then

$$\|\psi_T U *_R f; X^{s,b}\| \leq CT^{1-b+b'} \|f; X^{s,b'}\|. \quad (3.10)$$

For $b > 1/2$, it is clear that $X^{s,b} \subset \mathcal{C}(\mathbb{R}, H^s)$. This is no longer true if $b \leq 1/2$ and we shall need the following Lemma for that result.

Lemma 3.2. (cf. [6] Lemma 2.2). Let $f \in Y^s$, then $\int_0^t dt' U(t-t')f(t') \in \mathcal{C}(\mathbb{R}, H^s)$.

Remark:

1. In the case $k < l + 1$ we shall be able to take (3.10) with $b, b_1 < 1/2$ and suitable $0 < c, c_1 < 1/2$ to estimate the following terms

- i) $n_\pm u, n_\pm Q, Q^2u$ and u in $X_1^{k, -c_1}$,
- ii) $\omega|u|^2$ and $\omega(QRe(u))$ in $X_2^{l, -c}$.

Furthermore, in order to get solution in $\mathcal{C}([0, T], H^k \times H^l \times H^{l-1})$, we also need to estimate the terms on Y_1^k and Y_2^l respectively. (Y_j corresponds to the symbol ϕ_j , $j = 1, 2$)

2. In the limit case $k = l + 1$ we shall be forced to take $b_1 = 1/2$. The estimates for $Q^2 u$, $Q n_{\pm}$, $n_{\pm} u$ and u in Y_1^k is also needed for the proof of the local well-posedness.
3. We are allowed to assume that u and n_{\pm} have compact support in t by using additional cutoffs inside f in (3.5) and consider the equation.

$$u(t) = \psi_1(t) U(t) u_0 - \psi_T(t) \int_0^t dt' U(t-t') f(\psi_{2T}(t') u(t')). \quad (3.11)$$

For the effect of those factors in the spaces $X^{s,b}$, we refer [6] Lemma 2.5.

The rest of this Section will be organized as follows: In section 3.2, we will prove the estimates for linear terms which are new in this context. The estimates for nonlinear terms were proved completely in [6] and we will recall them in section 3.3. Finally, we give the final step of the proof of Theorem 1.

3.2. Estimates for linear terms. We may assume that $n_{\pm} \in X_2^{l,b}$ and $u \in X_1^{k,b_1}$ then they can be rewritten in the form

$$\begin{aligned} \widehat{n_{\pm}} &= \langle \xi \rangle^{-l} \langle \tau \pm |\xi| \rangle^{-b} \widehat{v}, \\ \widehat{u} &= \langle \xi \rangle^{-k} \langle \tau + |\xi|^2 \rangle^{-b_1} \widehat{w}, \\ \widehat{\bar{u}} &= \langle \xi \rangle^{-k} \langle \tau - |\xi|^2 \rangle^{-b_1} \widehat{\bar{w}}, \end{aligned}$$

where $v, w \in L^2(\mathbb{R}^2)$. Then

$$\begin{aligned} \widehat{n_{\pm} Q}(\xi, \tau) &= \mathcal{F}_x(Q(x) \mathcal{F}_{yt}(n_{\pm}))(\xi_1) \\ &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{n_{\pm}}(\xi_1 - \xi'_1, \xi_2, \tau) d\xi'_1 \\ &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-l} \\ &\quad \times \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^{-b} d\xi'_1, \\ \widehat{Q u}(\xi, \tau) &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-k} \\ &\quad \times \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} d\xi'_1, \end{aligned}$$

and

$$\begin{aligned} \widehat{|Q|^2 u}(\xi, \tau) &= \int \mathcal{F}_x\{|Q|^2\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-k} \\ &\quad \times \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} d\xi'_1. \end{aligned}$$

Furthermore, in order to estimate $\omega(Q \operatorname{Re}(u))$ we will rewrite it as $\omega(Q \frac{u + \bar{u}}{2})$, then we also need the following form

$$\begin{aligned} \widehat{Q \bar{u}}(\xi, \tau) &= \int \mathcal{F}_x\{Q\}(\xi'_1) \widehat{\bar{w}}(\xi_1 - \xi'_1, \xi_2, \tau) \langle (\xi_1 - \xi'_1, \xi_2) \rangle^{-k} \\ &\quad \times \langle \tau - |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} d\xi'_1. \end{aligned}$$

It is important in our arguments that $\mathcal{F}_x\{Q\}$ and $\mathcal{F}_x\{Q^2\}$ are positive functions. In order to estimate $n_{\pm} Q$ in $X_1^{k,-c_1}$, we take its scalar product with a generic function in X_1^{-k,c_1} with Fourier transform $\langle \xi \rangle^k \langle \tau + |\xi|^2 \rangle^{-c_1} \widehat{v}_1$ and $v_1 \in L^2$. The required estimate of $n_{\pm} Q$ in $X_1^{k,-c_1}$ then takes the form

$$|I_1| \leq CT^{\theta_1} \|v\|_2 \|v_1\|_2, \quad (3.12)$$

where

$$I_1 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_1(\xi, \tau) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^l \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + |\xi|^2 \rangle^{c_1}} d\xi d\tau d\xi'_1,$$

with the notation $\xi = (\xi_1, \xi_2)$.

Similarly, in order to estimate Q^2u in $X_1^{k, -c_1}$, the required estimate takes the form

$$|I_2| \leq CT^{\theta_2} \|w\|_2 \|v_2\|_2, \quad (3.13)$$

where

$$I_2 = \int \frac{\mathcal{F}_x\{Q^2\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_2(\xi, \tau) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau + |\xi|^2 \rangle^{c_1}} d\xi d\tau d\xi'_1,$$

with $v_2 \in L^2$. The required estimate for u in $X_1^{k, -c_1}$ is simpler than (3.13) since we do not have the term Q and the variable ξ'_1 .

Similarly, in order to estimate $\omega(QRe(u))$ in $X_2^{l, -c}$, we will estimate $\omega(Qu)$ and $\omega(Q\bar{u})$ in $X_2^{l, -c}$. We take its scalar product with a generic function in $X_2^{-l, c}$ with Fourier transform $\langle \xi \rangle^l \langle \tau \pm |\xi| \rangle^{-c} \widehat{v}_2$ and $v_2 \in L^2$. The required estimates for $\omega Re(Qu)$ then takes the form

$$|I_3| \leq CT^{\theta_3} \|w\|_2 \|v_2\|_2, \quad (3.14)$$

$$|I_4| \leq CT^{\theta_4} \|w\|_2 \|v_2\|_2, \quad (3.15)$$

where

$$I_3 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_2(\xi, \tau) |\xi| \langle \xi \rangle^l}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} d\xi d\tau d\xi'_1,$$

$$I_4 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_2(\xi, \tau) |\xi| \langle \xi \rangle^l}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau - |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} d\xi d\tau d\xi'_1.$$

We now consider the estimates for the linear terms in Y_1^k and Y_2^l . The estimates for Q^2u and u are similar, then we will only estimate Q^2u in Y_1^k . Similarly to the previous part, the required estimates for Q^2u in Y_1^k will take the form :

$$|I_5| \leq CT^{\theta_5} \|w\|_2 \|v_3\|_{L_x^2}, \quad (3.16)$$

where

$$I_5 = \int \frac{\mathcal{F}_x\{Q^2\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_3(\xi) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau + |\xi|^2 \rangle} d\xi d\tau d\xi'_1.$$

Estimate for Qn_{\pm} in Y_1^k :

$$|I_6| \leq CT^{\theta_6} \|v\|_2 \|v_3\|_{L_x^2}, \quad (3.17)$$

where

$$I_6 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_3(\xi) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^l \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + |\xi|^2 \rangle} d\xi d\tau d\xi'_1.$$

Estimate for $\omega(Qu)$ in Y_2^l :

$$|I_7| \leq CT^{\theta_7} \|w\|_2 \|v_3\|_{L_x^2}, \quad (3.18)$$

where

$$I_7 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_3(\xi) \langle \xi \rangle^l |\xi|}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle} d\xi d\tau d\xi'_1.$$

Estimate for $\omega(Q\bar{u})$ in Y_2^l :

$$|I_8| \leq CT^{\theta_8} \|w\|_2 \|v_3\|_{L_x^2}, \quad (3.19)$$

where

$$I_8 = \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_3(\xi) \langle \xi \rangle^l |\xi|}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau - |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle} d\xi d\tau d\xi'_1.$$

Because of the effect of cutoff function, in (3.12) - (3.19) we are allowed to assume that the functions $\mathcal{F}^{-1}(\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^{-b} \widehat{v})$, $\mathcal{F}^{-1}(\langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{-b_1} \widehat{w})$ will be supported in a region $|t| < CT$.

Preparing for the proofs of (3.12) - (3.17), we first recall the Strichartz estimate and some elementary inequalities which we will need.

Lemma 3.3. (cf. [6] Lemma 3.1) Let $b_0 > 1/2$, let $a \geq 0$, $a' \geq 0$, let $0 \leq \gamma \leq 1$. Assume in addition that $(1 - \gamma)a \leq b_0$ and $\gamma a \leq a'$. Let $0 < \eta \leq 1$ and define q and r by

$$2/q = 1 - \eta(1 - \gamma)a/b_0 \quad (3.20)$$

$$1 - 2/r = (1 - \eta)(1 - \gamma)a/b_0. \quad (3.21)$$

Let $v \in L^2$ be such that $\mathcal{F}^{-1}(\langle \tau + |\xi|^2 \rangle^{-a'} \widehat{v})$ has support in $|t| \leq CT$. Then

$$\left\| \mathcal{F}^{-1}(\langle \tau + |\xi|^2 \rangle^{-a} |\widehat{v}|); L^q L^r \right\| \leq CT^\theta \|v\|_2, \quad (3.22)$$

with $\theta \geq 0$. Note that $\theta = 0$ if and only if $a = 0$ or $\gamma = 0$.

Remark: In particular, in this paper, in order to estimate the linear terms we shall use only the special case $q = r = 2$, $\gamma = 1$ for both Schrödinger equation and wave equation.

Lemma 3.4. Let $\xi_1, \xi'_1, \xi_2, \tau \in \mathbb{R}$ and $\xi = (\xi_1, \xi_2)$, then there exists $C > 0$ such that these following estimates hold,

$$\langle \xi \rangle^2 \leq C (\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + \langle \tau + |\xi|^2 \rangle + \langle \xi'_1 \rangle), \quad (3.23)$$

$$\langle \xi \rangle^2 \leq C \left(\langle \tau \pm |\xi| \rangle + \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle + \langle \xi'_1 \rangle^2 \right), \quad (3.24)$$

$$\langle \xi \rangle \leq C \langle \xi'_1 \rangle \langle (\xi_1 - \xi'_1, \xi_2) \rangle \quad (3.25)$$

Proof. Proof of (3.23). By Cauchy-Schwarz inequality

$$\begin{aligned} & \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^2 + \langle \tau + |\xi|^2 \rangle^2 + \langle \xi'_1 \rangle^2 \\ &= 3 + (\tau \pm |(\xi_1 - \xi'_1, \xi_2)|)^2 + (\tau + |\xi|^2)^2 + \xi_1'^2 \\ &\geq 3 + (|\xi|^2 + |\xi'_1| \mp |(\xi_1 - \xi'_1, \xi_2)|)^2 / 3 \end{aligned}$$

and

$$\begin{aligned} & |(\xi_1 - \xi'_1, \xi_2)| \\ &= \sqrt{(\xi_1 - \xi'_1)^2 + \xi_2^2} \\ &\leq (|\xi_1 - \xi'_1| + |\xi_2|) \\ &\leq (|\xi_1| + |\xi'_1| + |\xi_2|). \end{aligned}$$

If $|\xi_1| + |\xi_2| \geq 4$ then

$$\begin{aligned} & |\xi|^2 + |\xi'_1| \\ &\geq \frac{(|\xi_1| + |\xi_2|)^2}{2} + |\xi'_1| \\ &\geq \frac{(|\xi_1| + |\xi_2|)^2}{4} + (|\xi_1| + |\xi_2| + |\xi'_1|) \end{aligned}$$

Therefore,

$$|\xi|^2 + |\xi'_1| \mp |(\xi_1 - \xi'_1, \xi_2)| \geq \frac{(|\xi_1| + |\xi_2|)^2}{4},$$

and

$$\begin{aligned} & \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^2 + \langle \tau + |\xi|^2 \rangle^2 + \langle \xi'_1 \rangle^2 \\ & \geq 3 + (|\xi_1| + |\xi_2|)^4 / 48 \\ & \geq 3 + (\xi_1^2 + \xi_2^2)^2 / 48 \\ & > \langle \xi \rangle^4 / 96. \end{aligned}$$

If $|\xi_1| + |\xi_2| < 4$ or $|\xi| < 4$ then there exists a constant C large enough such that

$$\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^2 + \langle \tau + |\xi|^2 \rangle^2 + \langle \xi'_1 \rangle^2 \geq 3 > \langle \xi \rangle^4 / C.$$

Therefore, there exists $C > 0$ such that (3.23) holds. The proof of (3.24) is similar. Proof of (3.25),

$$\begin{aligned} & \langle \xi'_1 \rangle^2 \langle \xi_1 - \xi'_1, \xi_2 \rangle^2 \\ & = (1 + \xi_1'^2) (1 + (\xi_1 - \xi'_1)^2 + \xi_2^2) \\ & \geq 1 + \xi_1'^2 + (\xi_1 - \xi'_1)^2 + \xi_2^2 \\ & \geq 1 + \xi_1^2 / 2 + \xi_2^2 \\ & > \langle \xi \rangle^2 / 2, \end{aligned}$$

then (3.25) holds. \square

Lemma 3.5. (i) (cf. [6] Lemma 3.3 (i)) Let $y_1, y_2 \in \mathbb{R}$ and $z = y_1 - y_2$. Then for any $\nu > 1$

$$|z| \leq \nu |y_2| + \frac{\nu}{\nu-1} |y_1| \chi(|z| \geq \nu |y_2|) \chi\left(\frac{\nu}{\nu+1} \leq \frac{|z|}{|y_1|} \leq \frac{\nu}{\nu-1}\right), \quad (3.26)$$

where $\chi(\Omega)$ is the characteristic function of the set Ω .

(ii) Let $\xi_1, \xi_2, \xi'_1 \in \mathbb{R}$, $\xi = (\xi_1, \xi_2)$ and let $|\xi| \geq 2|\xi'_1|$. Then

$$\langle \xi \rangle^2 \leq C (\langle \xi'_1 \rangle + 2 \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + 2 \langle \tau + |\xi|^2 \rangle \chi(\mathcal{B})), \quad (3.27)$$

where

$$\mathcal{B} = \left((\tau, \xi) : \frac{1}{2} (|\xi|^2 - \frac{3}{2} |\xi|) \leq |\tau + |\xi|^2| \leq \frac{3}{2} (|\xi|^2 + \frac{3}{2} |\xi|) \right).$$

Proof. Proof of (i): See [6] lemma 3.3.

Proof of (ii): Using (i) with $z = |\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)|$, $y_1 = \tau + |\xi|^2$, $y_2 = \tau \pm |(\xi_1 - \xi'_1, \xi_2)|$ and $\nu = 2$, we have

$$\begin{aligned} & \left| |\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)| \right| \leq 2 |\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| \\ & + 2 |\tau + |\xi|^2| \chi\left(\frac{2}{3} \leq \frac{||\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)||}{\tau + |\xi|^2} \leq 2\right). \end{aligned} \quad (3.28)$$

Now we will take a close look on the set

$$\mathcal{A} = \left((\tau, \xi) : \frac{2}{3} \leq \frac{||\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)||}{\tau + |\xi|^2} \leq 2 \right).$$

By using triangle inequality and the fact that $|\xi| \geq 2|\xi'_1|$, we have

$$\begin{aligned} & \left| |\xi|^2 \mp |(\xi_1 - \xi'_1, \xi_2)| \right| \leq |\xi|^2 + |(\xi_1 - \xi'_1, \xi_2)| \\ & \leq |\xi|^2 + |\xi| + |\xi'_1| \\ & \leq |\xi|^2 + \frac{3}{2} |\xi|, \end{aligned}$$

and

$$\begin{aligned} \|\xi\|^2 \mp |(\xi_1 - \xi'_1, \xi_2)| &\geq |\xi|^2 - |(\xi_1 - \xi'_1, \xi_2)| \\ &\geq |\xi|^2 - \frac{3}{2}|\xi|. \end{aligned}$$

Therefore,

$$\mathcal{A} \subset \mathcal{B}.$$

Then (3.28) follows

$$\begin{aligned} |\xi|^2 &\leq |(\xi_1 - \xi'_1, \xi_2)| + 2|\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| + 2|\tau + |\xi|^2|_{\mathcal{X}(\mathcal{B})} \\ &\leq |\xi| + |\xi'_1| + 2|\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| + 2|\tau + |\xi|^2|_{\mathcal{X}(\mathcal{B})}. \end{aligned}$$

Set $A = |\xi'_1| + 2|\tau \pm |(\xi_1 - \xi'_1, \xi_2)|| + 2|\tau + |\xi|^2|_{\mathcal{X}(\mathcal{B})} + \frac{1}{4}$, then

$$|\xi| \leq \sqrt{A} + \frac{1}{2},$$

and then

$$|\xi|^2 \leq A + \sqrt{A} + \frac{1}{4}.$$

Therefore, (3.27) holds with suitable constant C . \square

Remark: (3.27) will be used to cancel the logarithmic singularities which will appear in (3.17).

In the next Lemma, we will prove (3.12), (3.13), (3.14) and (3.15).

Lemma 3.6. *Let $k, l \geq 0$ and $k = l + \epsilon$ with $0 < \epsilon \leq 1$. Let b, b_1, c, c_1 satisfy:*

(i) *If $0 < \epsilon < 1$.*

$$\frac{\epsilon}{2} < b < \frac{1}{2}, \quad \frac{1-\epsilon}{2} < b_1 < \frac{1}{2} \quad (3.29)$$

$$\frac{\epsilon}{2} \leq c_1 < \frac{1}{2}, \quad \frac{1-\epsilon}{2} \leq c < \frac{1}{2}. \quad (3.30)$$

(ii) *If $\epsilon = 1$.*

$$\frac{1}{2} < b < 1, \quad b_1 = \frac{1}{2} \quad (3.31)$$

$$0 < c < 1 - b, \quad c_1 = \frac{1}{2} \quad (3.32)$$

then the estimates (3.12), (3.13), (3.14) and (3.15) hold and $\theta_1, \theta_2, \theta_3, \theta_4 > 0$.

Proof. i) *Proof of (3.12).*

By using symbol inequalities (3.23), (3.25) and the fact that $k = l + \epsilon$, we have

$$\begin{aligned} |I_1| &= \left| \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{v}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_1(\xi, \tau) \langle \xi \rangle^k}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^l \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}} d\xi d\tau d\xi'_1 \right| \\ &\leq C \int \frac{|\langle \xi'_1 \rangle^{l+1/2} \mathcal{F}_x\{Q\}(\xi'_1) |\widehat{v}| |\widehat{v}_1|}{\langle \xi'_1 \rangle^{1/2} \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}} \\ &\quad \times (\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + \langle \tau + \xi^2 \rangle + \langle \xi'_1 \rangle)^{\epsilon/2}. \end{aligned}$$

Let us note that

$$\begin{aligned} &\frac{(\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle + \langle \tau + \xi^2 \rangle + \langle \xi'_1 \rangle)^{\epsilon/2}}{\langle \xi'_1 \rangle^{1/2} \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}} \\ &\approx \frac{\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^{\epsilon/2} + \langle \tau + \xi^2 \rangle^{\epsilon/2} + \langle \xi'_1 \rangle^{\epsilon/2}}{\langle \xi'_1 \rangle^{1/2} \langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^b \langle \tau + \xi^2 \rangle^{c_1}}. \end{aligned}$$

Then by subtracting $\epsilon/2$ from one of $(b, c_1, 1/2)$ and using the hypothesis $b > \epsilon/2$ ($> 1/2$ if $\epsilon = 1$), we only need to consider the terms of the following form, with $\alpha > 0$, $\beta \geq 0$ and $s \geq 0$.

$$\begin{aligned} \bar{I}_1 &= \int \frac{|\langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1)| |\widehat{v}| |\widehat{v}_1|}{\langle \tau \pm |(\xi_1 - \xi'_1, \xi_2)| \rangle^\alpha \langle \tau + |\xi|^2 \rangle^\beta} d\xi d\tau d\xi'_1, \\ &\leq \int \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1) \left(\langle \tau \pm |\xi| \rangle^{-\alpha} \widehat{v} \right) (\xi_1 - \xi'_1) \Big| d\xi'_1 \right) \\ &\quad \times \left(\langle \tau + |\xi|^2 \rangle^{-\beta} |\widehat{v}_1| \right) d\xi d\tau. \end{aligned}$$

By using Plancherel identity, Hölder inequality, we have

$$\begin{aligned} \bar{I}_1 &\leq \left\| \mathcal{F}^{-1} \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1) \left(\langle \tau \pm |\xi| \rangle^{-\alpha} \widehat{v} \right) (\xi_1 - \xi'_1) \Big| d\xi'_1 \right) \right\|_2 \\ &\quad \times \left\| \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\beta} |\widehat{v}_1| \right) \right\|_2. \end{aligned}$$

The first term on the right hand side is rewritten as follow

$$\begin{aligned} &\mathcal{F}^{-1} \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x \{Q\}(\xi'_1) \left(\langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right) (\xi_1 - \xi'_1, \xi_2, \tau) \Big| d\xi'_1 \right) \\ &= \mathcal{F}^{-1} \left(\int |\mathcal{F}_x (\langle D_x \rangle^s Q) (\xi'_1)| \left| \left(\langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right) (\xi_1 - \xi'_1, \xi_2, \tau) \right| d\xi'_1 \right) \\ &= \mathcal{F}^{-1} \left(|\mathcal{F}_x (\langle D_x \rangle^s Q)| * \left| \langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right| (\xi_1) \right) \\ &= \mathcal{F}_{yt}^{-1} \left(\mathcal{F}_x^{-1} |\mathcal{F}_x (\langle D_x \rangle^s Q)| \cdot \mathcal{F}_x^{-1} \left| \langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right| \right) \\ &= \mathcal{F}_{yt}^{-1} \left((\langle D_x \rangle^s Q) \cdot \mathcal{F}_x^{-1} \left| \langle \tau + \phi(\xi) \rangle^{-b} \widehat{v} \right| \right) \\ &= (\langle D_x \rangle^s Q(x)) \mathcal{F}^{-1} \left(\langle \tau + \phi(\xi) \rangle^{-b} |\widehat{v}| \right), \end{aligned}$$

We used the fact that $\mathcal{F}_x (\langle D_x \rangle^s Q) (\xi_1) = \langle \xi_1 \rangle^s Q(\xi_1) > 0$.

With $\phi(\xi) = \pm|\xi|$. This implies

$$\begin{aligned} \bar{I}_1 &\leq C \left\| (\langle D_x \rangle^s Q(x)) \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\alpha} |\widehat{v}| \right) \right\|_2 \\ &\quad \times \left\| \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\beta} |\widehat{v}_1| \right) \right\|_2 \\ &\leq C \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\alpha} |\widehat{v}| \right) \right\|_2 \times \left\| \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\beta} |\widehat{v}_1| \right) \right\|_2 \\ &\leq CT^{\theta_1} \|v\|_2 \|v_1\|_2. \end{aligned}$$

We used the fact that $\langle D_x \rangle^s Q(x) \in L_x^\infty$ for any $s \geq 0$, and $\theta_1 > 0$ comes from the Strichartz estimate (3.22) with $\alpha > 0$. Therefore, (3.12) holds even in the limit case $c_1 = 1/2$.

ii) The proof of (3.13) is easier than (3.12), because we only need to use the estimate (3.25) to remove $\langle \xi \rangle^k$ in the numerator and using that $\mathcal{F}_x(Q^2) > 0$, $\langle D_x \rangle^s Q^2(x) \in L_x^\infty$ for any $s \geq 0$.

iii) *Proof of (3.14) and (3.15).* We only prove (3.14), (3.15) is treated similarly, because, with $a > 0$

$$\begin{aligned} &\left\| \langle \tau - \xi^2 \rangle^{-a} \widehat{w}(\tau, \xi); L_\tau^2 L_\xi^2 \right\| \\ &= \left\| \langle \tau - \xi^2 \rangle^{-a} \widehat{w}(-\tau, -\xi); L_\tau^2 L_\xi^2 \right\| \\ &= \left\| \langle \tau + \xi^2 \rangle^{-a} \widehat{w}(\tau, \xi); L_\tau^2 L_\xi^2 \right\|. \end{aligned}$$

By using symbol inequalities (3.24), (3.25) and the fact that $k = l + \epsilon$, we have

$$\begin{aligned}
|I_3| &= \left| \int \frac{\mathcal{F}_x\{Q\}(\xi'_1) \widehat{w}(\xi_1 - \xi'_1, \xi_2, \tau) \widehat{v}_2(\xi, \tau) |\xi| \langle \xi \rangle^l}{\langle \xi_1 - \xi'_1, \xi_2 \rangle^k \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} d\xi d\tau d\xi'_1 \right| \\
&\leq C \int \frac{|\langle \xi'_1 \rangle^k \mathcal{F}_x\{Q\}(\xi'_1)| |\widehat{w}| |\widehat{v}_2| \langle \xi \rangle^{1-\epsilon}}{\langle \tau + |(\xi_1 - x'_1, x_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} d\xi d\tau d\xi'_1 \\
&\leq C \int \frac{|\langle \xi'_1 \rangle^k \mathcal{F}_x\{Q\}(\xi'_1)| |\widehat{w}| |\widehat{v}_2|}{\langle \tau + |(\xi_1 - x'_1, x_2)|^2 \rangle^{b_1} \langle \tau \pm |\xi| \rangle^c} \\
&\quad \times \left(\langle \tau \pm |\xi| \rangle + \langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle + \langle \xi'_1 \rangle^2 \right)^{\frac{1-\epsilon}{2}}.
\end{aligned}$$

Subtracting $\frac{1-\epsilon}{2}$ from one of $(b_1, c, 1/2)$ and using the fact that $b_1 > \frac{1-\epsilon}{2}$, we only need to consider the terms of the following form with $\alpha > 0$, $\beta \geq 0$ and $s > 0$.

$$\bar{I}_3 = \int \frac{|\langle \xi'_1 \rangle^s \mathcal{F}_x\{Q\}(\xi'_1)| |\widehat{w}| |\widehat{v}_2|}{\langle \tau + |(\xi_1 - \xi'_1, \xi_2)|^2 \rangle^\alpha \langle \tau \pm |\xi| \rangle^\beta} d\xi d\tau d\xi'_1.$$

Similarly as in the previous part, we have

$$\begin{aligned}
\bar{I}_3 &\leq \int \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x\{Q\}(\xi'_1) \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} \widehat{w} \right) (\xi_1 - \xi'_1) | d\xi'_1 \right) \\
&\quad \times \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) d\xi d\tau \\
&\leq \left\| \mathcal{F}^{-1} \left(\int |\langle \xi'_1 \rangle^s \mathcal{F}_x\{Q\}(\xi'_1) \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} \widehat{w} \right) (\xi_1 - \xi'_1) | d\xi'_1 \right) \right\|_2 \\
&\quad \times \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) \right\|_2 \\
&\leq C \left\| \langle (D_x)^s Q(x) \rangle \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} |\widehat{w}| \right) \right\|_2 \\
&\quad \times \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) \right\|_2 \\
&\leq C \left\| \mathcal{F}^{-1} \left(\langle \tau + |\xi|^2 \rangle^{-\alpha} |\widehat{w}| \right) \right\|_2 \times \left\| \mathcal{F}^{-1} \left(\langle \tau \pm |\xi| \rangle^{-\beta} |\widehat{v}_2| \right) \right\|_2 \\
&\leq CT^{\theta_3} \|w\|_2 \|v_2\|_2.
\end{aligned}$$

Therefore, (3.14) holds with $\theta_3 > 0$. \square

Remark: In the case $\epsilon = 1$, we take $b > 1/2$ since we need a positive power of T on (3.12) and also on (3.17).

We now prove (3.16) and (3.17).

Lemma 3.7. *Let $k, l \geq 0$ and $k = l + \epsilon$ with $0 < \epsilon \leq 1$. Let b, b_1 satisfy:*

(i) *If $0 < \epsilon < 1$.*

$$\frac{\epsilon}{2} < b < \frac{1}{2}, \quad \frac{1-\epsilon}{2} < b_1 < \frac{1}{2}, \quad (3.33)$$

(ii) *If $\epsilon = 1$.*

$$\frac{1}{2} < b < 1, \quad b_1 = 1/2 \quad (3.34)$$

then (3.16), (3.17), (3.18) and (3.19) hold and $\theta_5, \theta_6, \theta_7, \theta_8 > 0$.

Proof. i) If $0 < \epsilon < 1$ then the proofs of (3.16), (3.17), (3.18) and (3.19) are very similar as the proofs of (3.12), (3.13), (3.14) and (3.15). It remains to prove two inequalities of the form

$$\left\| \langle \tau + |\xi|^2 \rangle^{-1/2-\delta} \widehat{v}_3(\xi); L_\xi^2 L_\tau^2 \right\| \leq C \|v_3\|_{L_x^2}$$

and

$$\left\| \langle \tau + \pm|\xi| \rangle^{-1/2-\delta} \widehat{v}_3(\xi); L_\xi^2 L_\tau^2 \right\| \leq C \|v_3\|_{L_x^2},$$

where $-1/2 - \delta = 1 - \epsilon/2$ or $1 - (1 - \epsilon)/2$. Therefore $\delta > 0$ and the inequalities hold. The positive power of T comes from the Strichartz estimate for remaining term.

ii) If $\epsilon = 1$ we only consider (3.17), because in (3.16) I_5 does not depend on l . In (3.18) and (3.19) we can use (3.25). We separate the integration region of I_6 into two subregions:

Region $|\xi| \leq 2|\xi'_1|$: The proof is the same as in part i) for (3.16) since we can eliminate the term $\langle \xi \rangle^k$.

Region $|\xi| \geq 2|\xi'_1|$: By using (3.27), it remains to estimate the following term

$$\begin{aligned} & \left\| \langle \tau + |\xi|^2 \rangle^{-1/2} \widehat{v}_3(\xi) \chi \left(\frac{1}{2}(|\xi|^2 - \frac{3}{2}|\xi|) \leq |\tau + |\xi|^2| \leq \frac{3}{2}(|\xi|^2 + \frac{3}{2}|\xi|) \right); L_\xi^2 L_\tau^2 \right\| \\ & \leq C(\ln(3))^{1/2} \|v_3\|_{L_x^2}. \end{aligned}$$

Therefore (3.17) holds, and $\theta_6 > 0$ comes from Strichartz estimate for the remaining term. \square

3.3. Estimates for nonlinear terms. *Remark:* In the following Lemmas, we are allowed to assume that n_\pm and u have compact support in time.

Lemma 3.8. (*Estimate for $n_\pm u$ in $X_1^{k, -c_1}$*) (cf. [6] Lemma 3.4). Let $b_0 > 1/2$ and $0 < b, c_1, b_1 \leq b_0 < b + c_1 + b_1 - c_0$, with $0 < c_0 \leq \text{Min}(b, c_1, b_1)$. And

$$\begin{aligned} 2b_0 &< b + c_1 + b_1, \\ l &\geq 0, |k| \leq l + 2c_0. \end{aligned}$$

Then

$$\left\| n_\pm u; X_1^{k, -c_1} \right\| \leq CT^\theta \left\| n_\pm; X_2^{l, b} \right\| \left\| u; X_1^{k, b_1} \right\|, \quad (3.35)$$

holds, with $\theta > 0$.

Lemma 3.9. (*Estimate for $\omega|u|^2$ in $X_2^{l, -c}$*) (cf. [6] Lemma 3.5). Let $b_0 > 1/2$ and let $0 < c, b_1 \leq b_0 < c + 2b_1 - \bar{c}_0$ with $0 < \bar{c}_0 \leq \text{Min}(c, b_1)$. And

$$\begin{aligned} 2b_0 &< c + 2b_1 \\ 2k &\geq l + 1, \quad k \geq l + 1 - 2\bar{c}_0, \quad k \geq 0. \end{aligned}$$

Then

$$\left\| \omega|u|^2; X_2^{l, -c} \right\| \leq CT^\theta \left\| u; X_1^{k, b_1} \right\|^2 \quad (3.36)$$

holds, with $\theta > 0$.

Lemma 3.10. (*Estimate for $n_\pm u$ in Y_1^k*) (cf. [6] Lemma 3.6). Let $b_0 > 1/2$, let $0 < c_1 < 1/2$ and let $0 < b, b_1 \leq b_0 < b + c_1 + b_1 - c_0$, with $0 < c_0 \leq \text{min}(b, 1/2, b_1)$. And

$$\begin{aligned} l &\geq 0, \quad |k| \leq l + 2c_0, \\ 2b_0 &< b + c_1 + b_1. \end{aligned}$$

Then

$$\left\| n_\pm u; Y_1^k \right\| \leq CT^\theta \left\| n_\pm; X_2^{l, b} \right\| \left\| u; X_1^{k, b_1} \right\| \quad (3.37)$$

holds, with $\theta > 0$.

Lemma 3.11. (*Estimate for $\omega|u|^2$ in Y_2^l*) (cf. [6] Lemma 3.7). Let $b_0 > 1/2$, let $0 < c < 1/2$ and let $0 < b_1 \leq b_0 < c + 2b_1 - \bar{c}_0$ where $0 < \bar{c}_0 \leq \text{Min}(1/2, b_1)$. And

$$\begin{aligned} 2b_0 &< c + 2b_1 \\ 2k &\geq l + 1, \quad k \geq l + 1 - 2\bar{c}_0. \end{aligned}$$

Then

$$\|\omega|u|^2; Y_2^l\| \leq T^\theta \left\| u; X_1^{k, b_1} \right\|^2 \quad (3.38)$$

holds, with $\theta > 0$.

Proof of Theorem 1

Proof. We finish the proof of Theorem 1 by combining the estimates for the linear and nonlinear terms in Section 3.2, 3.3 and the contraction argument described in Section 3.1.

- (i) If $k < l + 1$, by Lemma 3.1, Lemma 3.2, Lemma 3.6 (i) and Lemma 3.7 (i), It is sufficient to find $1/2 > c, c_1 \geq 0$ such that estimates (3.35), (3.36), (3.37) and (3.38) hold. In Lemmas 3.8, 3.9, 3.10 and 3.11 we choose

$$\begin{aligned} c_1 &= \text{Min}(1 - b_1, 1/2) & c &= \text{Min}(1 - b, 1/2) \\ c_0 &= \text{Min}(b, 1 - b_1, b_1) & \bar{c}_0 &= \text{Min}(1 - b, b_1, 1/2). \end{aligned} \quad (3.39)$$

Then with b, b_1 close enough to $1/2$, the conditions in Lemmas 3.8, 3.9, 3.10 and 3.11 will be fulfilled.

Remember that in this case, we only need estimates (3.16), (3.17), (3.18), (3.19), (3.10) and (3.11) to get solution in $(u, n, \partial_t n) \in \mathcal{C}(H^k \times H^l \times H^{l-1})$.

- (ii) If $k = l + 1$, by Lemma 3.1 and Lemma 3.7 (ii) we are forced to take $b_1 = 1/2$ and $b > 1/2$. For estimates (3.12)-(3.15) we use Lemma 3.6 (ii), for estimates (3.16)-(3.17) we use Lemma 3.7 (ii), for estimates (3.35)-(3.36) we use the same choice of (c, c_1, c_0, \bar{c}_0) as in (3.39), for (3.37) we choose $c_0 = c_1 = 1/2$. If we choose b close enough to $1/2$ then all the required estimates for linear and nonlinear terms hold.

Furthermore, in order to extend the solution to $\mathcal{C}(H^k \times H^l \times H^{l-1})$ we also need (3.18), (3.19) and (3.38) given by Lemmas 3.7 (ii) and 3.11.

Up to now we proved the well-posedness of the cut off integral equation (3.6)-(3.7), for the proof of the independence of the cut off function we refer to [6]. \square

4. SCHOCHET-WEINSTEIN METHOD

4.1. Zakharov system as a dispersive perturbation of a symmetric hyperbolic system. In this section, we will follow the method of Schochet and Weinstein ([18]) to rewrite Zakharov system (1.1)-(1.2) as a dispersive perturbation of a symmetric hyperbolic system.

First, we define some auxiliary functions

$$V = -\frac{1}{\lambda} \Delta^{-1} \nabla(n_t) \quad (4.1)$$

$$P = n + |u|^2, \quad (4.2)$$

then (1.1)-(1.2) will become

$$iu_t + \Delta u + |u|^2 u - Pu = 0 \quad (4.3)$$

$$P_t + \lambda \nabla \cdot V - (|u|^2)_t = 0 \quad (4.4)$$

$$V_t + \lambda \nabla P = 0. \quad (4.5)$$

Multiplying (4.3) by \bar{u} and taking the imaginary part of the resulting equation to get

$$(|u|^2)_t = i(\bar{u}\Delta u - u\Delta\bar{u}). \quad (4.6)$$

Next, we take the gradient of (4.3) and get

$$i\nabla u_t + \Delta\nabla u + |u|^2\nabla u + (u\nabla\bar{u} + \bar{u}\nabla u)u - P\nabla u - u\nabla P = 0. \quad (4.7)$$

Now let $\sqrt{2}u = F + iG$ and $\sqrt{2}\nabla u = H + iL$. Then, the use of (4.6) in (4.4) leads to the following system equivalent to (4.3)-(4.5), (4.7)

$$P_t + \lambda\nabla \cdot V + F\nabla \cdot L - G\nabla \cdot H = 0 \quad (4.8)$$

$$V_t + \lambda\nabla P = 0 \quad (4.9)$$

$$F_t + \frac{1}{2}(F^2 + G^2)G - PG = -\Delta G \quad (4.10)$$

$$G_t - \frac{1}{2}(F^2 + G^2)F + PF = \Delta F \quad (4.11)$$

$$H_t - G\nabla P + \frac{1}{2}(F^2 + G^2)L + (FH + GL)G - PL = -\Delta L \quad (4.12)$$

$$L_t + F\nabla P - \frac{1}{2}(F^2 + G^2)H - (FH + GL)F + PH = \Delta H. \quad (4.13)$$

Introducing the 9 - component vector function $U = (P, V, F, G, H, L)^T$, equations (4.8)-(4.13) can be rewritten as follows

$$U_t + \sum_{j=1}^2 (A^j(U) + \lambda C^j)U_{x_j} + B(U)U = K\Delta U, \quad (4.14)$$

where A^j and C^j are symmetric 9×9 matrices, K is a antisymmetric matrix. C^j and K are constant matrices.

In the next section, we will use the above argument to rewrite (1.6)-(1.7) in a similar form.

4.2. Perturbed Zakharov system as a dispersive perturbation of a symmetric hyperbolic system. In Section 3, we already considered the perturbation of (1.1)-(1.2) by $Q(x) = 2\sqrt{2}/(e^x + e^{-x})$ that is (1.6)-(1.7). In this section, we will consider the perturbation of (1.1)-(1.2) by $e^{it}Q(x)$. This trick will make the calculation simpler by using the previous calculation.

Furthermore, in order to make computation transparent, we will denote $(e^{it}Q(x), -|Q(x)|^2)$ by (ϕ, ϕ_1) ¹ and change the notation of spatial variables from (x, y) to (x_1, x_2) . Then, the perturbed system will have the form

$$i\partial_t(u + \phi) + \Delta(u + \phi) = (n + \phi_1)(u + \phi) \quad (4.15)$$

$$\frac{1}{\lambda^2}\partial_t^2(n + \phi_1) - \Delta(n + \phi_1) = \Delta|u + \phi|^2 \quad (4.16)$$

(ϕ, ϕ_1) is a 1-D solution of (1.1) - (1.2), then if we denote

$$V_r = -\frac{1}{\lambda}\Delta^{-1}\nabla(\partial_t\phi_1) = -\frac{1}{\lambda}\Delta^{-1}(\partial_{x_1}\partial_t\phi_1, 0),$$

$$P_r = \phi_1 + |\phi|^2$$

then (ϕ, Q_r, V_r) is also a solution of (4.3)-(4.5).

We denote that

$$\tilde{V} = V + V_r = -\frac{1}{\lambda}\Delta^{-1}\nabla(\partial_t(n + \phi_1))$$

$$\tilde{P} = P + P_r = (n + \phi_1) + |u + \phi|^2,$$

¹These notations are not the same as in Section 3 where ϕ and ϕ_1 denote something different.

then because of (4.15)-(4.16), $(u + \phi, \tilde{P}, \tilde{V})$ is a solution of (4.3)-(4.5). Now applying the argument in the previous section, if we denote

$$\begin{aligned}\sqrt{2}\phi &= F_r + iG_r, & \sqrt{2}\nabla\phi &= H_r + iL_r \\ \sqrt{2}(u + \phi) &= \tilde{F} + i\tilde{G} = F + F_r + i(G + G_r) \\ \sqrt{2}\nabla(u + \phi) &= \tilde{H} + i\tilde{L} = H + H_r + i(L + L_r),\end{aligned}$$

then $(P_r, V_r, F_r, G_r, H_r, L_r)^T$ and $(\tilde{P}, \tilde{V}, \tilde{F}, \tilde{G}, \tilde{H}, \tilde{L})^T$ are the solutions of (4.8)-(4.13). Therefore, we can get a system solved by $(P, V, F, G, H, L)^T$ by eliminating the terms only depend on $(P_r, V_r, F_r, G_r, H_r, L_r)$, that is the following system

$$P_t + \lambda\nabla \cdot V + (F + F_r)\nabla \cdot L - (G + G_r)\nabla \cdot H + \mathcal{R}_1 = 0 \quad (4.17)$$

$$V_t + \lambda\nabla P = 0 \quad (4.18)$$

$$F_t + \mathcal{R}_2 = -\Delta G \quad (4.19)$$

$$G_t + \mathcal{R}_3 = \Delta F \quad (4.20)$$

$$H_t - (G + G_r)\nabla P + \mathcal{R}_4 = -\Delta L \quad (4.21)$$

$$L_t + (F + F_r)\nabla P + \mathcal{R}_5 = \Delta H. \quad (4.22)$$

Where the residuals are

$$\begin{aligned}\mathcal{R}_1 &= F\nabla \cdot L_r - G\nabla \cdot H_r \\ \mathcal{R}_2 &= \frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)G + \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)G_r - \tilde{P}G - PG_r \\ \mathcal{R}_3 &= -\frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)F - \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)F_r + \tilde{P}F + PF_r \\ \mathcal{R}_4 &= \frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)L + \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)L_r \\ &\quad + (\tilde{F}\tilde{H} + \tilde{G}\tilde{L})G + (FH + GL + FH_r + F_rH + GL_r + G_rL)G_r \\ &\quad - PL_r - P_rL - PL \\ \mathcal{R}_5 &= -\frac{1}{2}(\tilde{F}^2 + \tilde{G}^2)H - \frac{1}{2}(F^2 + G^2 + 2FF_r + 2GG_r)H_r \\ &\quad - (\tilde{F}\tilde{H} + \tilde{G}\tilde{L})F - (FH + GL + F_rH + FH_r + G_rL + GL_r)F_r \\ &\quad + PH + P_rH + PH_r.\end{aligned}$$

Therefore, we now can rewrite (4.15)-(4.16) as a dispersive perturbation of a symmetric hyperbolic system

$$U_t + \sum_{j=1}^2 (A^j(U) + B^j(\phi) + \lambda C^j)U_{x_j} + (D^1(U) + D^2(\phi))U = K\Delta U, \quad (4.23)$$

where $U = (P, V, F, G, H, L)^T$, A^j, B^j, C^j are symmetric 9×9 matrices, K is an anti-symmetric 9×9 matrix.

$$A^1(U) = \begin{pmatrix} 0 & 0_{1 \times 4} & -G & 0 & F & 0 \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ -G & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix},$$

$$\begin{aligned}
 B^1(\phi) &= \begin{pmatrix} 0 & 0_{1 \times 4} & -G_r & 0 & F_r & 0 \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ -G_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix}, \\
 A^2(U) &= \begin{pmatrix} 0 & 0_{1 \times 4} & 0 & -G & 0 & F \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ -G & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix}, \\
 B^2(\phi) &= \begin{pmatrix} 0 & 0_{1 \times 4} & 0 & -G_r & 0 & F_r \\ 0_{4 \times 1} & 0_{4 \times 4} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ -G_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ 0 & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \\ F_r & 0_{1 \times 4} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} & 0_{1 \times 1} \end{pmatrix}.
 \end{aligned}$$

Now, instead of studying the Cauchy problem for the perturbed Zakharov system (4.15)-(4.16) we will study the Cauchy problem for (4.23).

In [18], without the appearance of $B^j(\phi)$ and $D^2(\phi)$, the authors proved the existence of the solution of (4.23) in H^s with $s > 2$, by modifying the method for quasilinear symmetric hyperbolic systems (see [10], [13]). In our situation, the non decay at infinity of ϕ will lead to some difficulties and will reduce the result to the case of integer order Sobolev space. In particular, in order to use the symmetry of $B^j(\phi)$ we will need the validation of Leibniz rule.

Theorem 3. *Let $s > 2$, $s \in \mathbb{N}$ and $U_0 \in (H^s)^9$, then there exists $T = T(\|U_0\|_{(H^s)^9})$ such that the equation (4.23) has a unique solution*

$$U \in L^\infty([0, T], (H^s(\mathbb{R}^2))^9),$$

with $U(0) = U_0$.

Proof. The proof of the existence proceeds via a classical iteration scheme. We first regularize the initial data by taking a family of self-adjoint regularization operators J_ϵ as following.

We choose $j \in C_0^\infty(\mathbb{R}^2)$, $\text{supp } j \subset \{X = (x_1, x_2) \in \mathbb{R}^2; |X| < 1\}$, $\int j = 1$ and set $j_\epsilon(X) = \epsilon^{-2}j(X/\epsilon)$. Then, we define $J_\epsilon u \in C^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$ by

$$J_\epsilon u = j_\epsilon * u.$$

Set $\epsilon_k = 2^{-k}$, $U_0^k = J_{\epsilon_k} U_0$. We construct a local solution of (4.23) by considering the iteration scheme :

$$U^0(x_1, x_2, t) = U_0^0(x_1, x_2), \quad (4.24)$$

$$U_t^{k+1} + \sum_{j=1}^2 (A^j(U^k) + B^j(\phi) + \lambda C^j) U_{x_j}^{k+1} + (D^1(U^k) + D^2(\phi)) U^{k+1} = K \Delta U^{k+1}, \quad (4.25)$$

$$U^{k+1}(x_1, x_2, 0) = U_0^{k+1}(x_1, x_2). \quad (4.26)$$

Since $\langle \xi \rangle^s \sim \langle \xi_1 \rangle^s + \langle \xi_2 \rangle^s$, $\forall s \geq 0$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we have $\|\cdot\|_{H^s} \sim \|J_{x_1}^s \cdot\|_{L^2} + \|J_{x_2}^s \cdot\|_{L^2}$, where $J_{x_1} = \mathcal{F}^{-1} \langle \xi_1 \rangle \mathcal{F}$, $J_{x_2} = \mathcal{F}^{-1} \langle \xi_2 \rangle \mathcal{F}$.

Therefore, in order to estimate $\|U^{k+1}\|_{H^s}$ we will estimate $\|J_{x_1}^s U^{k+1}\|_{L^2}$ and $\|J_{x_2}^s U^{k+1}\|_{L^2}$ separately.

Estimate for $\|J_{x_1}^s U^{k+1}\|_{L^2}$.

Applying $J_{x_1}^s$ to (4.25), we get

$$\begin{aligned} \partial_t(J_{x_1}^s U^{k+1}) + \sum_{j=1}^2 J_{x_1}^s \left((A^j(U^k) + B^j(\phi))U_{x_j}^{k+1} \right) + \sum_{j=1}^2 \lambda C^j J_{x_1}^s U_{x_j}^{k+1} \\ + J_{x_1}^s \left((D^1(U^k) + D^2(\phi))U^{k+1} \right) = K \Delta J_{x_1}^s U^{k+1}. \end{aligned}$$

Or

$$\begin{aligned} \partial_t(J_{x_1}^s U^{k+1}) + \sum_{j=1}^2 (A^j(U^k) + B^j(\phi))J_{x_1}^s U_{x_j}^{k+1} + \sum_{j=1}^2 \lambda C^j J_{x_1}^s U_{x_j}^{k+1} + (D^1(U^k) + D^2(\phi))J_{x_1}^s U^{k+1} \\ + \sum_{j=1}^2 [J_{x_1}^s, A^j(U^k) + B^j(\phi)]U_{x_j}^{k+1} + [J_{x_1}^s, D^1(U^k) + D^2(\phi)]U^{k+1} = K \Delta J_{x_1}^s U^{k+1}, \end{aligned}$$

with notation for commutators $[F, G] = FG - GF$.

We multiply (4.28) by $J_{x_1}^s U^{k+1}$ and integrating in \mathbb{R}^2 , with integration by parts and using the symmetry of A^j and B^j we obtain

$$\begin{aligned} \frac{1}{2} \partial_t \|J_{x_1}^s U^{k+1}\|_{L^2}^2 \\ + \sum_{j=1}^2 \left(-\frac{1}{2} \left\langle (A^j(U_{x_j}^k) + B^j(\phi_{x_j}))J_{x_1}^s U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle + \left\langle [J_{x_1}^s, A^j(U^k) + B^j(\phi)]U_{x_j}^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \right) \\ + \left\langle (D^1(U^k) + D^2(\phi))J_{x_1}^s U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle + \left\langle [J_{x_1}^s, D^1(U^k) + D^2(\phi)]U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ = 0. \end{aligned} \tag{4.27}$$

Note that $\langle \cdot, \cdot \rangle$ denotes $L^2(\mathbb{R}^2)$ scalar product and $\langle \cdot, \cdot \rangle_1$ denotes $L^2(\mathbb{R})$ scalar product with respect to x_1 .

Before going to estimate (4.27), we need the following lemma (see [11]. B2).

Definition 4.1. We say that a Fourier multiplier $\sigma(D)$ is of order s ($s \in \mathbb{R}$) and write $\sigma \in \mathcal{S}^s$ if $\xi \in \mathbb{R}^d \mapsto \sigma(\xi) \in \mathcal{C}$ is smooth and satisfies

$$\forall \xi \in \mathbb{R}^d, \forall \beta \in \mathbb{N}^d, \quad \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta| - s} |\partial^\beta \sigma(\xi)| < \infty.$$

Lemma 4.1. Let $t_0 > d/2$, $s \geq 0$ and $\sigma \in \mathcal{S}^s$. If $f \in H^s(\mathbb{R}^d) \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{s-1}(\mathbb{R}^d) \cap H^{t_0}(\mathbb{R}^d)$ then

$$\|[\sigma(D), f]g\|_{L^2} \leq C(\sigma) (\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}} \|g\|_{L^\infty}), \tag{4.28}$$

where $C(\sigma)$ depends only on σ .

First, using (4.28) with $d = 2$, $\sigma(D) = J_{x_1}^s$, Sobolev embedding inequality and note that $D^1(U)$ contains linear and quadratic terms on U and $A^j(U)$ depends linearly on U , we get

$$\begin{aligned} \sum_{j=1}^2 \left\langle [J_{x_1}^s, A^j(U^k)]U_{x_j}^{k+1}, J_{x_1}^s U^{k+1} \right\rangle + \left\langle [J_{x_1}^s, D^1(U^k)]U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ \lesssim \left(\|U^k\|_{H^s} + \|U^k\|_{H^s}^2 \right) \|J_{x_1}^s U^{k+1}\|_{L^2}^2 \\ \lesssim \left(\|U^k\|_{H^s} + \|U^k\|_{H^s}^2 \right) \|U^{k+1}\|_{H^s}^2 \end{aligned} \tag{4.29}$$

Now using (4.28) with $d = 1$, $\sigma(D) = J_{x_1}^s$ and Sobolev embedding inequality, we have

$$\begin{aligned} & \sum_{j=1}^2 \left\langle [J_{x_1}^s, B^j(\phi)] U_{x_j}^{k+1}, V^{k+1} \right\rangle_1 + \left\langle [J_{x_1}^s, D^2(\phi)] U^{k+1}, V^{k+1} \right\rangle_1 \\ & \lesssim C(\|\phi\|_{W^{s,\infty}}, \|\phi_1\|_{W^{s,\infty}}) (\|U_{x_2}^{k+1}\|_{H_{x_1}^{s-1}} + \|U_{x_2}^{k+1}\|_{L_{x_1}^\infty} + \|U^{k+1}\|_{H_{x_1}^s}) \|J_{x_1}^s U^{k+1}\|_{L_{x_1}^2} \\ & \leq C(\|\phi\|_{W^{s,\infty}}, \|\phi_1\|_{W^{s,\infty}}) (\|U_{x_2}^{k+1}\|_{H_{x_1}^{s-1}} + \|J_{x_1}^s U_{x_2}^{k+1}\|_{L_{x_1}^2} + \|U^{k+1}\|_{H_{x_1}^s}) \|J_{x_1}^s U^{k+1}\|_{L_{x_1}^2}. \end{aligned} \quad (4.30)$$

Then, integrating both side of (4.30) in \mathbb{R} with respect to x_2 we obtain

$$\begin{aligned} & \sum_{j=1}^2 \left\langle [J_{x_1}^s, B^j(\phi)] U_{x_j}^{k+1}, V^{k+1} \right\rangle + \left\langle [J_{x_1}^s, D^2(\phi)] U^{k+1}, V^{k+1} \right\rangle \\ & \lesssim C(\|\phi\|_{W^{s,\infty}}, \|\phi_1\|_{W^{s,\infty}}) \|U^{k+1}\|_{H^s}^2. \end{aligned} \quad (4.31)$$

It is not hard to see that

$$\begin{aligned} & \sum_{j=1}^2 -\frac{1}{2} \left\langle (A^j(U_{x_j}^k) + B^j(\phi_{x_j})) J_{x_1}^s U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ & \lesssim (\|U^k\|_{H^s} + \|\nabla\phi\|_{W^{s,\infty}}) \|U^{k+1}\|_{H^s}^2. \end{aligned} \quad (4.32)$$

Combining (4.27), (4.29), (4.31) and (4.32) we get

$$\frac{1}{2} \partial_t \|J_{x_1}^s U^{k+1}\|_{L^2} \leq C(1 + \|U^k\|_{H^s} + \|U^k\|_{H^s}^2) \|U^{k+1}\|_{H^s}^2. \quad (4.33)$$

Estimate for $\|J_{x_2}^s U^{k+1}\|_{L^2}$.

We do similarly as previous step but note that ϕ does not depend on x_2 then

$$\begin{aligned} & \partial_t (J_{x_2}^s U^{k+1}) + \sum_{j=1}^2 (A^j(U^k) + B^j(\phi)) J_{x_2}^s U_{x_j}^{k+1} + \sum_{j=1}^2 \lambda C^j J_{x_2}^s U_{x_j}^{k+1} \\ & + \sum_{j=1}^2 [J_{x_2}^s, A^j(U^k)] U_{x_j}^{k+1} + (D^1(U^k) + D^2(\phi)) J_{x_2}^s U^{k+1} \\ & + [J_{x_2}^s, D^1(U^k)] U^{k+1} \\ & = K \Delta J_{x_2}^s U^{k+1}, \end{aligned}$$

Multiplying the above expression with $J_{x_2}^s U^{k+1}$, integrating in \mathbb{R}^2 , using integral by parts and the symmetry of A^j, B^j ($j = 1, 2$). We get

$$\begin{aligned} & \frac{1}{2} \partial_t \|J_{x_2}^s U^{k+1}\|_{L^2}^2 \\ & + \sum_{j=1}^2 \left(-\frac{1}{2} \left\langle (A^j(U_{x_j}^k) + B^j(\phi_{x_j})) J_{x_2}^s U^{k+1}, J_{x_2}^s U^{k+1} \right\rangle + \left\langle [J_{x_2}^s, A^j(U^k)] U_{x_j}^{k+1}, J_{x_2}^s U^{k+1} \right\rangle \right) \\ & + \\ & + \left\langle (D^1(U^k) + D^2(\phi)) J_{x_2}^s U^{k+1}, J_{x_2}^s U^{k+1} \right\rangle + \left\langle [J_{x_2}^s, D^1(U^k)] U^{k+1}, J_{x_1}^s U^{k+1} \right\rangle \\ & = 0. \end{aligned} \quad (4.34)$$

We can see that there is no “ ϕ ” term in the commutator part of (4.34), that make it is easier to estimate than (4.27). Indeed, we also have

$$\frac{1}{2} \partial_t \|J_{x_2}^s U^{k+1}\|_{L^2} \leq C(1 + \|U^k\|_{H^s} + \|U^k\|_{H^s}^2) \|U^{k+1}\|_{H^s}^2. \quad (4.35)$$

Thanks to (4.33) and (4.35) we have

$$\frac{1}{2} \|U^{k+1}\|_{H^s}^2 \leq C \left(1 + \|U^k\|_{H^s} + \|U^k\|_{H^s}^2\right) \|U^{k+1}\|_{H^s}^2. \quad (4.36)$$

We now consider a closed ball in $(H^s(\mathbb{R}^2))^9$

$$\mathcal{B} = \{U \in (H^s(\mathbb{R}^2))^9 : \|U\|_{H^s}^2 \leq \|U_0\|^2 + 1\}.$$

Therefore by combining (4.36), Gronwall inequality and induction method we can prove that

$$\|U^k(t)\|_{H^s}^2 \leq \|U_0\|_{H^s}^2 + 1, \quad (4.37)$$

for $t \in [0, T]$ and $T = T(\|U_0\|_{H^s})$.

Equation (4.25) then give us the uniform bound on U_t^k

$$\|U_t^k\|_{L^\infty([0, T]; (H^{s-2})^9)} \leq C(\|U_0\|_{H^s}).$$

Sequence $\{U^k\}$ is uniformly bounded in $L^\infty([0, T], (H^s)^9)$ then there exists a subsequence $\{U^{k_n}\}$ which converges to $U \in L^\infty([0, T], (H^s)^9)$ weak star. Furthermore, by Aubin-Lions theorem (Lemma 5.1), there is a subsequence of $\{U^{k_n}\}$ (with the same notation) converging strongly in $L^\infty([0, T]; (H_{loc}^{s-2})^9)$ and therefore in $L^\infty([0, T]; (H_{loc}^{s-\delta})^9)$ with $0 < \delta < 2$. This allows to take the limit in the nonlinear terms of (4.23).

The uniqueness of solution of (4.23) is done by using energy method for the difference of two solutions since the dispersive part contributes nothing to the energy. \square

Proof of Theorem 2.

Proof. Theorem 2 is a consequence of Theorem 3 as follows:

First, set $u = (1/\sqrt{2})(F + iG)$ then (4.18)-(4.20) implies

$$i\partial_t(u + \phi) + \Delta(u + \phi) + |u + \phi|^2(u + \phi) - \tilde{P}(u + \phi) = 0, \quad (4.38)$$

and

$$\tilde{V}_t + \lambda \nabla \tilde{P} = 0. \quad (4.39)$$

Where $\tilde{P} = P + P_r$ and $\tilde{V} = V + V_r$.

Next, From (4.19)-(4.21) we can derive an L^2 energy estimate of $W = (\nabla F - H, \nabla G - L)$, that implies

$$\|W(t)\|_{L^2}^2 \leq e^{CT} \|W(0)\|_{L^2}^2 = 0.$$

Therefore, $\nabla u = \frac{1}{\sqrt{2}}(H + iL)$. it leads to

$$(|u + \phi|^2)_t = i(\overline{u + \phi})\Delta(u + \phi) - (u + \phi)\Delta(\overline{u + \phi}),$$

and

$$\tilde{P}_t + \lambda \nabla \cdot \tilde{V} - (|u + \phi|^2)_t = 0.$$

Set $\tilde{n} = \tilde{P} - |u + \phi|^2$ we have that

$$\tilde{n}_t + \lambda \nabla \cdot \tilde{V} = 0. \quad (4.40)$$

Combining (4.39) and (4.40), we obtain (4.16), it is also clear that (4.38) implies (4.15). Estimate (1.12) follows similarly. \square

5. A WEAK CONVERGENCE RESULT

We consider the following family of system labeled by the parameter λ

$$i\partial_t u_\lambda - u_\lambda + \Delta u_\lambda = n_\lambda u_\lambda - Q^2 u_\lambda + Q n_\lambda \quad (5.1)$$

$$\frac{1}{\lambda^2} \partial_t^2 n_\lambda - \Delta n_\lambda = \Delta(|u_\lambda|^2) + 2\Delta(Q \operatorname{Re}(u_\lambda)) \quad (5.2)$$

$$\partial_t n_\lambda + \nabla \cdot f_\lambda = 0, \quad (5.3)$$

with initial data given by

$$u_\lambda(0) = u_0, \quad n_\lambda(0) = n_0, \quad f_\lambda(0) = f_0. \quad (5.4)$$

We study the behaviour of the solution (u_λ, n_λ) when λ tends to ∞ , that is the theorem

Theorem 4. *Let $s > 2$, $s \in \mathbb{N}$ and let $(u_\lambda, n_\lambda, f_\lambda) \in L^\infty(0, T, H^{s+1}) \times L^\infty(0, T, H^s) \times L^\infty(0, T, H^s)$ be the local solution of the system (5.1)-(5.3) given by Theorem 2 with $t \in [0, T]$ and initial data (5.4). Then as $\lambda \rightarrow \infty$, (u_λ, n_λ) converges to $(u, -|u|^2)$ in $L^\infty(0, T, H^1) \times L^\infty(0, T, L^2)$ weak star, where u is the unique solution of the perturbed nonlinear Schrödinger equation (1.13) with initial data $u_0 \in H^s$.*

We use a classical compactness method and we will follow the ideas of H. Added and S. Added in [2]. Before giving the proof of theorem 4 we want to recall the two following lemma in [12].

Lemma 5.1. *(Aubin-Lions's theorem) Let B_0, B, B_1 be three reflexive Banach spaces. Assume that B_0 is compactly embedded in B and B is continuously embedded in B_1 . Let*

$$W = \left\{ V \in L^{p_0}(0, T, B_0), \frac{\partial V}{\partial t} \in L^{p_1}(0, T, B_1) \right\}, \quad T < \infty, \quad 1 < p_0, p_1 < \infty.$$

W is a Banach space with norm

$$\|V\|_W = \|V\|_{L^{p_0}(0, T, B_0)} + \|V_t\|_{L^{p_1}(0, T, B_1)}.$$

Then the embedding $W \hookrightarrow L^{p_0}(0, T, B)$ is compact.

Lemma 5.2. *Let Ω be an open set of \mathbb{R}^n and let $g, g_\varepsilon \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, such that*

$$g_\varepsilon \rightarrow g \quad \text{a.e in } \Omega \quad \text{and} \quad \|g_\varepsilon\|_{L^p(\Omega)} \leq C.$$

Then $g_\varepsilon \rightarrow g$ weakly in $L^p(\Omega)$.

Proof of Theorem 4

Proof. (i) By using Theorem 2, we have that the quantities $\|u_\lambda\|_{L^\infty(0, T, H^1)}$, $\|n_\lambda\|_{L^\infty(0, T, L^2)}$ and $\|(1/\lambda)f_\lambda\|_{L^\infty(0, T, L^2)}$ are bounded uniformly in λ . So, some subsequence of $(u_\lambda, n_\lambda, (1/\lambda)f_\lambda)$ (still denoted by λ) has a weak limit (u, n, w) . More precisely

$$u_\lambda \rightarrow u \text{ in } L^\infty(0, T, H^1) \text{ weak star (w.s.)}, \quad (5.5)$$

$$n_\lambda \rightarrow n \text{ in } L^\infty(0, T, L^2) \text{ w.s.} \quad (5.6)$$

and

$$(1/\lambda)f_\lambda \rightarrow w \text{ in } L^\infty(0, T, L^2) \text{ w.s.} \quad (5.7)$$

Let us note that the following maps are continuous

$$\begin{aligned} H^1(\mathbb{R}^2) &\rightarrow L^4(\mathbb{R}^2) \\ u &\mapsto u, \end{aligned} \quad (5.8)$$

$$\begin{aligned} H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) &\rightarrow H^{-1}(\mathbb{R}^2) \\ (u, v) &\mapsto uv, \end{aligned} \quad (5.9)$$

$$\begin{aligned} H^1(\mathbb{R}^2) &\rightarrow H^{-1}(\mathbb{R}) \\ u &\mapsto Q^2u, \end{aligned} \quad (5.10)$$

$$\begin{aligned} H^1(\mathbb{R}^2) &\rightarrow L^2(\mathbb{R}^2) \\ u &\mapsto Qu \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} L^2(\mathbb{R}^2) &\rightarrow H^{-1}(\mathbb{R}^2) \\ v &\mapsto Qv. \end{aligned} \quad (5.12)$$

Since Q and ∇Q are bounded.

It then follows from (5.5)-(5.12) that the quantities $\|Q^2u_\lambda\|_{L^\infty(0,T,H^{-1})}$, $\|Qn_\lambda\|_{L^\infty(0,T,H^{-1})}$, $\|\Delta u_\lambda\|_{L^\infty(0,T,H^{-1})}$, $\| |u_\lambda|^2 \|_{L^\infty(0,T,L^2)}$, $\|QRe(u_\lambda)\|_{L^\infty(0,T,L^2)}$ and $\|n_\lambda u_\lambda\|_{L^\infty(0,T,H^{-1})}$ are bounded uniformly in λ . So it can be assumed that

$$n_\lambda u_\lambda \rightarrow g \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.}, \quad (5.13)$$

$$|u_\lambda|^2 \rightarrow h \text{ in } L^\infty(0, T, L^2) \text{ w.s.}, \quad (5.14)$$

$$QRe(u_\lambda) \rightarrow QRe(u) \text{ in } L^\infty(0, T, L^2) \text{ w.s.}, \quad (5.15)$$

$$Q^2u_\lambda \rightarrow Q^2u \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.}, \quad (5.16)$$

$$Qn_\lambda \rightarrow Qn \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.}, \quad (5.17)$$

and

$$\Delta u_\lambda \rightarrow \Delta u \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.} \quad (5.18)$$

Finally, taking into account (5.5), (5.13), (5.16), (5.17) and (5.18), equation (5.1) implies that

$$\partial_t u_\lambda \rightarrow \partial_t u \text{ in } L^\infty(0, T, H^{-1}) \text{ w.s.} \quad (5.19)$$

Using the above results, this proof will be complete if we establish that

$$n + |u|^2 + 2QRe(u) = 0 \quad (5.20)$$

and

$$g + u|u|^2 + 2QuRe(u) = 0. \quad (5.21)$$

(ii) *Proof of (5.20)*. Let Ω be any bounded subdomain of \mathbb{R}^2 . We notice that the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact, and for any Banach space X ,

$$\text{the embedding } L^\infty(0, T, X) \hookrightarrow L^2(0, T, X) \text{ is continuous.} \quad (5.22)$$

Hence, according to (5.22), (5.5), (5.19) and applying Lemma 5.1 to $B_0 = H^1(\Omega)$, $B = L^4(\Omega)$, $B_1 = H^{-1}(\Omega)$, $p_0 = p_1 = 2$, it implies that some subsequence of u_λ (on Ω) (also denoted by λ) converges strongly to u on the domain Ω in $L^2(0, T, L^4(\Omega))$. So we can assume that

$$u_\lambda \rightarrow u \text{ strongly in } L^2(0, T, L^4_{loc}(\mathbb{R}^2)) \quad (5.23)$$

and thus,

$$u_\lambda \rightarrow u \text{ almost everywhere in } (t, x, y) \in (0, T, \mathbb{R}^2) \quad (5.24)$$

Then, using Lemma 5.2, (5.14) and (5.24) implies that $h = |u|^2$.

Taking into account that $\|(1/\lambda)f_\lambda\|_{L^\infty(0,T,L^2)}$ is bounded uniformly in λ , equations (5.2) and (5.3) follow that

$$\nabla(n_\lambda + |u_\lambda|^2 + 2QRe(u_\lambda)) \rightarrow 0 \text{ in } \mathcal{D}'(0, T, L^2).$$

So $\nabla(n + |u|^2 + 2QRe(u)) = 0$. Since $n + |u|^2 + 2QRe(u) \in L^\infty(0, T, L^2)$, it follows that

$$n + |u|^2 + 2QRe(u) = 0.$$

(iii) *Proof of (5.21)*. We shall prove that $n_\lambda u_\lambda \rightarrow -u|u|^2 - 2QRe(u)$ in $L^2(0, T, H^{-1})$ w.s, which combined with (5.13) and (5.22) will ensure that $g = -u|u|^2 - 2QRe(u)$. First, let ψ be some test function in $L^2(0, T, H^1)$ vanishing out of a compact set $\Omega \in \mathbb{R}^2$. Then

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} (n_\lambda u_\lambda + u|u|^2 + 2QRe(u)) \psi \, dx \, dt \\ &= \int_0^T \int_\Omega n_\lambda (u_\lambda - u) \psi \, dx \, dt + \int_0^T \int_\Omega (n_\lambda + |u|^2 + 2QRe(u)) u \psi \, dx \, dt. \end{aligned}$$

Let $I_\lambda^1(\psi)$ and $I_\lambda^2(\psi)$ denote the first and the second integral on the right side respectively. Then

$$|I_\lambda^1(\psi)| \leq \|n_\lambda\|_{L^\infty(0, T, L^2)} \|\psi\|_{L^2(0, T, L^4(\Omega))} \|u_\lambda - u\|_{L^2(0, T, L^4(\Omega))}.$$

Since Ω is bounded we deduce from (5.6) and (5.23) that $I_\lambda^1(\psi)$ goes to 0 when λ tends to ∞ .

Using Sobolev embedding theorem and Hölder inequality, we see that $\psi u \in L^1(0, T, L^2)$. Therefore, using that $n_\lambda \rightarrow -|u|^2 - 2QRe(u)$ in $L^\infty(0, T, L^2)$ w.s, it follows that $I_\lambda^2(\psi)$ goes to 0 when λ tends to ∞ .

Hence,

$$\int_0^T \int_{\mathbb{R}^2} (n_\lambda u_\lambda + u|u|^2 + 2QRe(u)) \psi \, dx \, dt \rightarrow 0$$

for every test function ψ then the proof is complete. \square

Conclusions:

1) We proved the local well-posedness in the usual Sobolev spaces for the 2-d Zakharov system perturbed by its 1-d soliton solution. This is the preliminary step to studying transverse stability (or instability) of that 1-d soliton under the 2-d Zakharov flow. Moreover, Theorem 4 shows that one obtains an NLS type equation in an appropriate limit.

It is also interesting to consider the same problem for the general vectorial Zakharov system.

2) Schochet-Weinstein method is also interesting if we consider a perturbation of a 3-d Zakharov system by its 2-d soliton. We think the same method should work there, too, since we only used the algebraic structure of the system and the fact that the soliton is smooth and bounded.

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