Why Renormalizable Noncommutative Quantum Field Theories

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Introduction

Noncommutative Field Theory

Covariant theories, self-dual theories

Ghost Hunting

Can we get back the REAL world?

Constructive Field Theory

Conclusions
Scales in the universe

Physical phenomena occur over a wide range of scales from the Planck scale
$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35} \text{m}$$
to the radius of the observable universe which in practice in comoving coordinates corresponds to about 45 billion light-years, hence around
$$4.4 \times 10^{26} \text{m} \text{ or better, } 2.7 \times 10^{61} \ell_P.$$
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The universe therefore is made of roughly 61 powers of 10 or 140 powers of $e$. 
We have a fair knowledge of the 45 biggest scales of the universe. But the fifteen to sixteen scales between $\ell_P$ and $2 \times 10^{-19}$ meters (about 1 Tev), make up the last true terra incognita of physics.
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After that treat no new spectacular advance of this type is planned on terra incognita, hence we have some time to deepen our theoretical and mathematical understanding.
Why R, NC, QFT?

R because renormalizable models are the ones who survive renormalization group flows. They are the generic building blocks of physics.

NC because one cannot measure length below the Planck scale: the energy of the necessary particles would create a black hole whose horizon would hide the measurement. Fundamental uncertainties of this type mean noncommutativity of space time itself at that scale.

QFT because that's what particle physics is about at the frontier of terra incognita.

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Renormalization

The first computations in quantum field theory always ended in infinite results. Computations that could be compared to actual experiments were developed only after finding a cure for these infinities, called renormalization. Renormalization (Dyson, Feynman, Schwinger, Tomonaga...) suppressed the QFT infinities by reformulating physical laws in terms of parameters that can be observed at low energy, also called renormalized parameters. Infinities were pushed into unobservable "bare" parameters. In spite of its initial success (explanation of the Lamb shift and of the anomalous magnetic factor $g^{-2}$ for the electron...), renormalization was not easily accepted:

- F. Dyson, one of its inventors, thought "this was just a trick we wanted to use for a few months until finding something better..."
- Mathematicians (until Alain...) considered renormalization as a kind of ill-defined recipe, which "pulled infinities under the rug..."
- In the 50's, L. Landau in the Soviet Union discovered an unexpected difficulty: the infinities thrown out through the door sort of reappeared through the window. This phenomenon has been called the "Landau ghost."
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The simplest renormalizable quantum field theory, $\phi^4$, is defined on Euclidean space $\mathbb{R}^4$ through its Schwinger functions which are the moments of the formal functional measure:

$$d\nu = \frac{1}{Z} e^{-\left(\frac{\lambda}{4!}\phi^4 - \frac{m^2}{2}\phi^2 - \frac{a}{2} \partial_\mu \phi \partial^\mu \phi\right)} D\phi,$$

where $\lambda$ is the coupling constant, positive in order for the theory to be stable; $m$ is the mass, which fixes the energy scale of the renormalized theory; $a$ is called the "wave function constant", in general fixed to 1; $Z$ is a normalization so that this measure should be a probability measure; $D\phi$ is a formal product $\prod_{x \in \mathbb{R}^4} d\phi(x)$ of Lebesgue measures at each point of $\mathbb{R}^4$. 
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$$C(p) = \frac{1}{(2\pi)^2} \frac{1}{p^2 + m^2}, \quad C(x, y) = \int_0^\infty d\alpha e^{-\alpha m^2} \frac{e^{-|x-y|^2/4\alpha}}{\alpha^2},$$

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The best method to study and understand renormalization in QFT is to cut the propagator into a sequence of slices à la Wilson for a multi-scale analysis:

$$C = \sum_i C^i, \quad C^i(x, y) = \int_{M^{-2i}}^{M^{-2(i+1)}} d\alpha \cdots \leq K M^{2i} e^{-cM^i \|x-y\|}$$
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This leads to a factor $M^{2i}$ per line and $M^{-4i}$ per vertex integration $\int d^4x$. 


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![Diagram of a subgraph](image)
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- **The power counting, dependent on the dimension $d$**: when such a subgraph is compared to a local vertex, the corresponding weight depends on the dimension and of the type of that subgraph. The sum over the gap between the internal and external energy may therefore either converge or diverge.
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For instance this bubble diverges logarithmically when $d = 4$ because there are two line factors $M^{2i}$ and a single internal integration $M^{-4i}$. 
The unavoidable ghost?

In the case of the $\phi^4_4$ theory, power counting tells us that only two and four point subgraphs do diverge (in the sense explained above). Four point functions diverge logarithmically and govern the renormalization of the coupling constant $\lambda$. The only one-loop graph which is one-particle irreducible is the bubble:

$$\lambda_{i} - \frac{1}{d\lambda_{i} di} = -\lambda_{i} + \beta(\lambda_{i})^2,$$

whose sign cannot be changed without losing stability. It corresponds to a quadratic one dimensional flow whose solution is well known to diverge in a finite time!

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![Bubble Diagram]

It governs the flow equation

$$-\lambda_{i-1} = -\lambda_i + \beta(-\lambda_i)^2, \quad \frac{d\lambda_i}{di} = +\beta(\lambda_i)^2, \quad (1.3)$$

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- Happy end, all the people in red in this page got the Nobel prize...
Noncommutative Geometry

Noncommutative geometry as you all know generalizes ordinary geometry. Ordinary smooth functions or observables form a commutative algebra under ordinary multiplication. For instance in classical mechanics observables are smooth functions on phase space. Quantum mechanics replaces this commutative algebra by a noncommutative algebra of operators, where Poisson brackets become commutators. This is the first physical example of noncommutative geometry. But direct space-time itself could be of this type; for instance at a certain scale new uncertainty relations could appear between length and width which would generalize Heisenberg's relations.
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where $\Theta^{\mu\nu}$ is an antisymmetric constant tensor which in the simplest case can be written as:

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The unique product (associative, but noncommutative) generated by these relations on Schwarz class functions is called the Moyal-Weyl, product and writes:

$$(f \star g)(x) = \int \frac{d^d k}{(2\pi)^d} d^d y f(x + \frac{1}{2} \Theta \cdot k) g(x + y) e^{i k \cdot y}$$

$$e^{i k x} \star e^{i k' x} = e^{-\frac{1}{2} \Theta^{ij} k_i k_j} e^{i (k + k') x}$$
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But one can probably also study fruitfully other strong field problems (such as quark confinement or the growth of charged polymers under strong magnetic fields) with NCQFT techniques. This may help to tackle problems which look untractable in the ordinary geometry language simply because they correspond in that geometry to non-perturbative and non-local effects.
The $\phi^4$ theory on Moyal $\mathbb{R}^4_\theta$

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Why Renormalizable Noncommutative Quantum Field Theories, Vienne, November 25 2007
Introduction  Noncommutative Field Theory  Covariant theories, self-dual theories  Ghost Hunting  Can we get back the REAL world?  Constructive Field Theory

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- The Moyal vertex can be computed explicitly. It is proportional to

$$\int \prod_{i=1}^4 d^4 x^i \phi(x^i) \delta(x_1 - x_2 + x_3 - x_4) \exp \left( 2i\theta^{-1} (x_1 \wedge x_2 + x_3 \wedge x_4) \right)$$
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Ultraviolet-infrared mixing

- Amplitudes of planar graphs remain unchanged.

\[
\lambda \int d^4 k e^{ik\mu}k^\nu \theta^{\mu\nu}k^2 + m^2 \propto \lambda \sqrt{m^2 \tilde{p}^2} K_1(\sqrt{m^2 \tilde{p}^2}) \sim p \to 0 \quad p^{-2}
\]

These divergences increase with perturbation order. All correlation functions are affected and diverge. The theory cannot be renormalized.
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\[ S = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu_0^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x) \]

where \( \tilde{x}_\mu = 2\Theta^{-1}_{\mu\nu} x^\nu \), is covariant under a symmetry \( p_\mu \leftrightarrow \tilde{x}_\mu \) called Langmann-Szabo symmetry and is renormalizable at every order in \( \lambda \)!
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Related models were explored in many papers by various subsets of an informal "MOV" group (Münster-Orsay-Vienna): Disertori, Gayral, de Goursac, Gurau, Grosse, Magnen, Malbouisson, R, Steinacker, Tanasa, Vignes-Tourneret, Wallet, Wohlgennant, Wulkenhaar...
The new propagator

\[ G(x, y) = \theta^4 \Omega \left( \pi \theta \right) \int_0^\infty d\alpha e^{-\mu^2 \theta^4 \Omega \alpha^2} \left( \frac{\sinh \alpha}{\alpha^2} \exp \left( -\Omega \theta \sinh \alpha \|x - y\|^2 - \Omega \theta \tanh \alpha \left( \|x\|^2 + \|y\|^2 \right) \right) \right). \]

(2.5)

and involves the Mehler kernel rather than the heat kernel. Multiscale analysis relies again on a slicing of that propagator according to a geometric sequence:

\[ G_i(x, y) = \int M^{-2(i-1)} M^{-2i} d\alpha \cdots \leq K M^2 i e^{-c_1 M^2 \|x - y\|^2 - c_2 M^{-2i} \left( \|x\|^2 + \|y\|^2 \right)} \]

The corresponding new renormalization group corresponds to a completely new mixture of the previous ultraviolet and infrared notions. Furthermore there exists only a half direction which is infinite for this RG.
The new propagator

The propagator for this theory is best understood through its parametric representation

\[ G(x, y) = \frac{\theta}{4\Omega} \left( \frac{\Omega}{\pi\theta} \right) \int_0^\infty d\alpha \, e^{-\frac{\mu_0^2}{4\Omega} \alpha} \]

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It follows again from the combination of two arguments. A new **Moyality principle** replaces **locality**:

This principle applies only to **planar graphs with a single external face**.
Planarity and Power Counting

There is also a new power counting. One must take into account cyclicity of the vertex hence graphs have ribbon lines.
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\[ \omega = \frac{d}{2} (F - EF) - L = \left( 2 - \frac{E}{2} \right) - 4g - 2 (EF - 1) \] si \( d = 4 \), où

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\begin{array}{c}
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Noncommutative Renormalization

Renormalization again follows from the combination of two arguments. Power counting tells us that only planar graphs with two and four external legs arriving on a single external face must be renormalized. The Moyality priniciple tells us that when the gap grows between internal and external lines in the sense of the new renormalization group slicing, these terms look like Moyal products. The corresponding counterterms are therefore of the form of the initial Lagrangian!
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Translation invariance

Some critics were addressed to the Grosse-Wulkenhaar model:

▶ The harmonic potential is just an infrared cutoff. No wonder that it cures the mixing!
▶ There is a preferred origin in this model which cannot therefore be related to true physics which is translation-invariant.

These critics are not fully justified:

▶ The harmonic potential is not a simple infrared cutoff among others, it is the one which makes the theory just renormalizable.
▶ The initial GW model indeed breaks translation invariance. However there may be scenarios to connect it to translation invariant models at lower energy.
▶ There exists other classes of models, which we now call covariant, whose propagator is similar to the one of GW, can be analyzed by the same methods and is indeed invariant under magnetic translations. Physical quantities (which are gauge-invariant) are then translation-invariant.
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Translation invariance

Some critics were addressed to the Grosse-Wulkenhaar model:

- The harmonic potential is just an infrared cutoff. No wonder that it cures the mixing!
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These critics are not fully justified:

- The harmonic potential is not a simple infrared cutoff among others, it is the one which makes the theory just renormalizable.

- The initial GW model indeed breaks translation invariance. However there may be scenarios to connect it to translation invariant models at lower energy.

- There exists other classes of models, which we now call covariant, whose propagator is similar to the one of GW, can be analyzed by the same methods and is indeed invariant under magnetic translations. Physical quantities (which are gauge-invariant) are then translation-invariant.
Covariant Models
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The Gross-Neveu Model

The noncommutative Gross-Neveu model on $\mathbb{R}^2$ is defined by

$$L = \frac{1}{2} \overline{\psi} \left( \frac{\partial}{\partial \theta} + \Omega / \tilde{x} + m \right) \psi + \sum_{a, b} \lambda_1 \overline{\psi}^a \psi^a \bar{\psi}^b \psi^b + \lambda_2 \overline{\psi}^a \psi^b \bar{\psi}^b \psi^a + \lambda_3 \overline{\psi}^a \bar{\psi}^b \psi^b \psi^a + \lambda_4 \overline{\psi}^a \bar{\psi}^b \psi^b \psi^a$$

where $\tilde{x} = 2(\Theta - 1)x$, $\Theta = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $a, b$ are color or spin indices.

This covariant model, more difficult to treat, was proved renormalizable by Vignes-Tourneret in the so-called orientable case ($\lambda_3 = \lambda_4 = 0$) and physical observables are indeed translation-invariant.

There exists also similar Bosonic models, which generalize the so-called Langmann-Szabo-Zarembo model.
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$\Rightarrow$ This covariant model, more difficult to treat, was proved renormalizable by Vignes-Tourneret in the so-called orientable case ($\lambda_3 = \lambda_4 = 0$) and physical observables are indeed translation-invariant.
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There exists also similar Bosonic models, which generalize the so-called Langmann-Szabo-Zarembo model.
Victory on the Landau ghost!
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Contrary to the initial expectations (Snyders, 1947...) noncommutativity of space-time does not prevent *infinities*. $\phi^4_4$ remains just renormalisable, with still infinitely many primitively divergent graphs. However there is a big improvement:
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Ghost hunting

This result was obtained in three steps:


Can we get back the REAL world? Constructive Field Theory...
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- **One loop**: H. Grosse and R. Wulkenhaar,
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- **Any loops**: M. Disertori, R. Gurau, J. Magnen and V. Rivasseau,
  Vanishing of Beta Function of Non Commutative $\Phi^4_4$ to all orders,
The ghost killing gun

We must now follow two main parameters under renormalization group flow, namely $\lambda$ and $\Omega$. At first order one finds

$$d\lambda_i \sigma_i \lambda^{2i},$$

$$d\Omega_i \sigma_i \lambda_i,$$

whose solution is:

$$\lambda \Omega$$

At $\Omega = 1$ (self-dual point) the field strength renormalization compensates the coupling constant renormalization so that $\lambda \phi^4$ remains invariant.
The ghost killing gun

We must now follow two main parameters under renormalization group flow, namely $\lambda$ and $\Omega$. At first order one finds

$$d\lambda_i/d\tau \approx a(1 - \Omega_i)\lambda_i^2,$$

$$d\Omega_i/d\tau \approx b(1 - \Omega_i)\lambda_i.$$
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We must now follow two main parameters under renormalization group flow, namely $\lambda$ and $\Omega$. At first order one finds

$$\frac{d\lambda_i}{di} \approx a(1 - \Omega_i)\lambda_i^2, \quad \frac{d\Omega_i}{di} \approx b(1 - \Omega_i)\lambda_i,$$

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We must now follow two main parameters under renormalization group flow, namely $\lambda$ and $\Omega$. At first order one finds

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![Graph showing the evolution of $\lambda$ and $\Omega$](image-url)
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![Graph showing the behavior of $\lambda$ and $\Omega$ with $i$.]

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The Ward identities at all loops

\[ Z(\eta, \bar{\eta}) = \int d\phi d\bar{\phi} \ e^{-\left(\phi X \phi + \phi X \bar{\phi} + A \bar{\phi} \phi + \frac{\lambda}{2} \phi \bar{\phi} \phi \bar{\phi} \right) + \bar{\phi} \eta + \bar{\eta} \phi} \]  

(4.6)
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Let \( U = e^{i M} \). One performs the "left" change of variables:

\[ \phi \rightarrow \phi^U = \phi U \quad \bar{\phi} \rightarrow \bar{\phi}^U = U \bar{\phi} \]  \hspace{1cm} (4.7)

which leads to

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and one obtains the Ward identities:
Dyson’s equations

- This is a classification of graphs (no combinatoric to check)!
- The second term has one ”left tadpole insertion”. It vanishes after mass renormalization.
The first term

\[ G_{(1)}^4 (0, m, 0, m) = \lambda C_0 m G^2 (0, m) \left( [G^2 (0, m)]^2 + G_{\text{ins}}^2 (0, 0; m) \right) \]  

(4.9)
The first term

\[ G^4_{(1)}(0, m, 0, m) = \lambda C_{0m} G^2(0, m) \left( [G^2(0, m)]^2 + G^2_{\text{ins}}(0, 0; m) \right) \tag{4.9} \]

The Ward identity gives:

\[
G^2_{\text{ins}}(0, 0; m) = \lim_{\zeta \to 0} G^2_{\text{ins}}(\zeta, 0; m) = \lim_{\zeta \to 0} \frac{G^2(0, m) - G^2(\zeta, m)}{\zeta} \\
= -\partial_L G^2(0, m) \rightarrow \\
G^4_{(1)}(0, m, 0, m) = \lambda [G^2(0, m)]^4 \frac{C_{0m}}{G^2(0, m)} [1 - \partial_L \Sigma(0, m)] \tag{4.10} \]
The third term

$$G_{(3)}^{4, \text{bare}} = C_0 m \sum_p G_{\text{ins}}^{4, \text{bare}}(p, 0; m, 0, m)$$

is obtained by opening the face $p$ of

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\[ G^{4, \text{bare}}_{(3)} = C_0m \sum_p G^{4, \text{bare}}_{\text{ins}}(p, 0; m, 0, m) \]

is obtained by opening the face \( p \) of \( G_{(3)}^{4, \text{bare}} = C_0m \sum_p G_{\text{ins}}^{4, \text{bare}}(p, 0; m, 0, m) \) But in the renormalized theory we must add the missing mass counterterm.

\[ G^{4}_{(3)} = C_0m \sum_p G^{4}_{\text{ins}}(0, p; m, 0, m) - C_0m(CT_{\text{missing}})G^{4}(0, m, 0, m) \] (4.11)
The third term

\[ G^4_{(3)} = C_0 m \sum_p G^4_{\text{bare}} (p, 0; m, 0, m) - C_0 m (CT_{\text{missing}}) G^4 (0, m, 0, m) \]  

(4.11)

But one has \( CT_{\text{missing}} = \Sigma^R (0, 0) - \Sigma (0, 0) \), hence one concludes:

\[ G^4_{(3)} (0, m, 0, m) = -C_0 m G^4 (0, m, 0, m) \frac{1}{G^2 (0, 0)} \frac{\partial \Sigma (0, 0)}{1 - \partial \Sigma (0, 0)} \]
Death of the ghost

One puts $G_3^4$ on the left side of Dyson’s equation:

$$G_4^4(1 + C_0m \frac{1}{G_2^2(0, 0) \frac{\partial \Sigma(0, 0)}{1 - \partial \Sigma(0, 0)}})$$

$$= \lambda[G_2^2(0, m)]^4 \frac{C_0m}{G_2^2(0, m)} [1 - \partial_L \Sigma(0, m)]$$  \hspace{1cm} (4.12)

and using $C_0m = 1/(m + A^{ren}); G_2^2(0, m) = 1/[m(1 - \partial \Sigma) + A^{ren}]$ one gets:

$$G_4^4(1 - \partial \Sigma + \frac{A^{ren}}{m + A^{ren}} \partial \Sigma) = \lambda[G_2^2]^4(1 - \partial \Sigma)^2(1 - \frac{m}{m + A^{ren}} \partial \Sigma)$$  \hspace{1cm} (4.13)

hence by simplifying, since red terms are equal, amputating, $\Gamma_4 = \lambda Z^2$ hence $\beta = 0!$
A beautiful mathematical toy?

The Grosse-Wulkenhaar model seems far from the ordinary world...
A beautiful mathematical toy?

The Grosse-Wulkenhaar model seems far from the ordinary world...
It is Euclidean, it has parameters $\theta$ and $\Omega$ which are unobserved at ordinary energies...
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The Grosse-Wulkenhaar model seems far from the ordinary world...
It is Euclidean, it has parameters $\theta$ and $\Omega$ which are unobserved at ordinary energies...

However its main advantage is that it is mathematically consistent. May be nature makes use of it in a certain way?
A possible scenario
A possible scenario

As a first step towards possible scenarios beyond the standard model that could make use of the GW model, we suggest that there could be a whole bunch of noncommutative worldlets each with a $\theta_i$ parameter. They would each contain a $GW_i$ model with its own $\Omega_i$ harmonic confining potential, all glued by a commutative space:
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A Lattice of wordlets?

At low energy only the ordinary world would remain visible... The zero modes of the worldlets would create an effective LOCAL interaction in the commutative world.
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At low energy only the ordinary world would remain visible... The zero modes of the worldlets would create an effective LOCAL interaction in the commutative world. That theory is somewhat similar to a lattice-regularized ordinary commutative theory, but it has no ultraviolet cutoff. As energy increases, bigger and bigger parts of the worldlets would appear.
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We currently try to elaborate this idea in collaboration with R. Gurau and A. Tanasa
Noncommutativity a possible alternative to supersymmetry?

One of the strongest arguments for *supersymmetry* is that it tames uv divergences. This was desired to explain the mass hierarchy problem. Also it helps the coupling constants of the $U(1)$, $SU(2)$ and $SU(3)$ gauge groups of the standard model to converge at a single scale:
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Noncommutativity a possible alternative to supersymmetry?

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![Graphs showing the behavior of coupling constants](image)

But the ghost killing mechanism is a new mechanism to tame ultraviolet divergences which is completely different from SUSY and deserves some careful study...
Constructive Field Theory

Constructive Field Theory = Resummation of Perturbative Field Theory. But is not obvious:

▶ Functional integral is good for global existence (\[ \int e^{-\lambda \phi^4} d\mu(\phi) \leq 1 \]) but bad to compute connected functions

▶ Connectivity is read easily on graphs, but they are too many (about \( n! \) at order \( n \))

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There are of course many:

- Covariant non-orientable Models
- Covariant self-dual Models
- Gauge Theories (Yang-Mills, Chern-Simons)
- Noncommutative Minkovski Space
- Condensed Matter Applications (Quantum Hall effect, polymer growth in strong magnetic field...)
- Non commutative "curved" geometries
- Constructive analysis of the GW model to be completed... Constructive dim Reg and dim Ren
- Possible link to between ordinary field theory and quantum gravity
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Some good news:

- There is more than compatibility between non-commutative geometry and quantum field theory.
- Quantum field theory on non-commutative space can be renormalised.
- Quantum field theory is better behaved on non-commutative space than on commutative space (no Landau ghost).
- It seems it can be fully built at the non-perturbative level.
- Renormalization group flows are modified when there is non commutativity of space-time.
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Thanks
Why Renormalizable Noncommutative Quantum Field Theories, Vienne, November 25 2007

Introduction  Noncommutative Field Theory  Covariant theories, self-dual theories  Ghost Hunting  Can we get back the REAL world?  Constructive Field Theory

Thanks

for your attention!