Nucleon Deformation from Lattice QCD

Antonios Tsapalis
National Technical University of Athens
School of Applied Mathematics and Physical Sciences
&
Hellenic Naval Academy

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Outline

• Introduction to Lattice QCD
• Hadronic States from the Lattice
• Limitations and Systematics
• Nucleon Deformation and $N \rightarrow \Delta(1232)$ Electromagnetic Transition Form Factors
  
  C. Alexandrou, G. Koutsou, H. Neff, J. Negele, W. Schroers, A. Tsapalis
  PRD 77, 085012 (2008)

• $\Delta(1232)$ Deformation and $\Delta$ Electromagnetic Form Factors
  
  C. Alexandrou, T. Korzec, G. Koutsou, Th. Leontiou, C. Lorce’, J. Negele,
  V. Pascalutsa, A. Tsapalis, M. Vanderhaeghen  
  arXiv:0810.3976

• Conclusions – Prospects

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Quarks and Gluons

\[ S[\psi_f, \bar{\psi}_f, A_\mu] = \int d^4x \left[ \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \sum_f \bar{\psi}_f (i \gamma_\mu D^\mu - m_f) \psi_f \right] \]

\[ F_{\mu\nu} = [D_\mu, D_\nu] \]

\[ D_\mu = \partial_\mu + igA_\mu \]

invariant under:

P, C, T,

local color SU(3) rotations

global flavour SU(N_f) rotations (approximate)

\[ g(q^2) \rightarrow 0 \quad q^2 \rightarrow \text{large} \quad \text{(asymptotic freedom)} \]

Perturbative techniques suffice for the description of high energy scattering processes
Lattice QCD

- Rotate to Euclidean time: \( t \rightarrow -i \tau \)

\[ |\Psi(\tau)\rangle = e^{-H \tau} |\Psi(0)\rangle \]

- Discretize space-time

\[ U_{x,\mu} = e^{igT^a A_\mu^a(x)} \]
Wilson formulation (1974)

\[ \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} \]

\[ 1 - \frac{1}{6} \{ \text{Tr}[U_1 U_2 U_3^+ U_4^+] + h.c. \} \]

Plaquette gauge action

\[ \bar{\psi}(i\gamma_\mu D^\mu - m)\psi \]

\[ (m + 4)\bar{\psi}_1 \psi_1 - \frac{1}{2} \{ \bar{\psi}_1 [(1 - \gamma_\mu) U_\mu + (1 + \gamma_\mu) U_\mu^+] \} \psi_2 \]

Wilson-Dirac operator \( D_W \)
### Lattice QCD Action

\[
S_{\text{gauge}} = \frac{6}{g^2} \sum_{\mu, \nu} \sum_x a^4 \left\{ 1 - \frac{1}{6} \text{Tr} [ P_{\mu\nu} (x) + P_{\mu\nu}^+ (x) ] \right\}
\]

\[
S_{\text{quark}} = \sum_{x, y} a^4 \overline{\psi}_x D_W (x, y) \psi_y
\]

\[
S_{\text{LQCD}} = S_{\text{gauge}} + S_{\text{quark}}
\]

**Invariant under local SU(3) transformations \( g_x \) on sites**

\[
U_{x, \mu} \to g_x U_{x, \mu} g_{x+\hat{\mu}}^+ \quad \psi_x \to g_x \psi_x \quad , \quad \overline{\psi}_x \to \overline{\psi}_x g_x
\]
Generate an ensemble of gauge fields \( \{U\} \) distributed with the Boltzmann weight

\[
Z = \int D\psi D\bar{\psi} DUE^{-\beta S_{\text{gauge}}[U]+\sum_{x,y}\bar{\psi}_x D_W\psi_y} = \int DUE^{-\beta S_{\text{gauge}}[U]} \det[D_W(U)]
\]

\( \beta = \frac{6}{g^2} \)

Calculate any n-point function of QCD

\[
\langle \hat{O} \rangle = \frac{1}{Z} \int DU D\psi D\bar{\psi} O[U,\psi_x,\bar{\psi}_y]e^{-\beta S_{\text{gauge}}[U]+\sum_{x,y}\bar{\psi}_x D_W\psi_y}
\]

\[
= \frac{1}{Z} \int DU O[U,D_W^{-1}]e^{-\beta S_{\text{gauge}}[U]} \det[D_W(U)]
\]

**stochastic solution (not simulation) of QCD**
Hadron masses in Lattice QCD

\[ B^p(x) = \varepsilon^{abc} [u^a(x)C\gamma_5 d^b(x)]u^c(x) \]

- construct interpolating field for hadron state
- generate a baryon at \( t=0 \)
- annihilate the baryon at time \( t \)
- measure the 2-pt function
- extract the energy from the exponential decay of the state in Euclidean time
Hadron 2-pt functions are products of quark propagators

nucleon interpolating field:

\[ B^p(x) = \varepsilon^{abc} [u^a(x)C\gamma_5d^b(x)]u^c(x) \]

\[ \langle \Omega \mid T(B_x\bar{B}_y) \mid \Omega \rangle = \varepsilon^{abc} \varepsilon^{a'b'c'} \]

\[ \langle D^{aa'}_{w^{-1}}(x, y) \otimes D^{bb'}_{w^{-1}}(x, y) \otimes D^{cc'}_{w^{-1}}(x, y) \rangle \]

expensive part: calculate quark propagator on each configuration

\[ \langle \Omega \mid T(\psi_x\bar{\psi}_y) \mid \Omega \rangle = D_{w^{-1}}(x, y) \]

Invert the \((3x4xL^3xT) \times (3x4xL^3xT)\) Wilson-Dirac matrix with standard linear system solvers (conjugate gradient etc..)
\[
\int d\vec{x} e^{-i\vec{p} \cdot \vec{x}} \langle \Omega | B(\vec{x}, \tau) \overline{B}(0) | \Omega \rangle = \\
\int d\vec{x} e^{-i\vec{p} \cdot \vec{x}} \langle \Omega | B(\vec{x}, \tau) | \sum_{\vec{q}} N(\vec{q}) \rangle \langle N(\vec{q}) | + | N'(\vec{q}) \rangle \langle N'(\vec{q}) | + \ldots | \overline{B}(0) | \Omega \rangle = \\
\sum_{\vec{q}} \int d\vec{x} e^{-i\vec{p} \cdot \vec{x}} e^{-E_N \tau} e^{i\vec{q} \cdot \vec{x}} \langle \Omega | B | N(\vec{q}) \rangle \langle N(\vec{q}) | B | \Omega \rangle + \text{excited states} = \\
Z_N^2 e^{-E_N \tau} + Z_N^2 e^{-E_N \tau} + \ldots = Z_N^2 e^{-E_N \tau} (1 + C_1 e^{-\Delta E \tau} + \ldots) \\
\]

use \[ \langle N(\vec{q}) | \overline{B}(\vec{x}, \tau) | \Omega \rangle = Z_N e^{-i\vec{q} \cdot \vec{x}} e^{-E_N \tau} \]

maximize the overlap \( Z_N \) to the state using "smeared" interpolating fields
$M_N a = 0.595 \pm 0.007$

Fixing one hadron mass to its physical value determines the lattice spacing $a$ – for the rest of the hadrons we get a prediction for the mass.
Limitations

physical answers emerge at the limit

Size of ensemble \{U\} \rightarrow \text{infinite} \quad O(100-1000)
Lattice volume \(L_a\) \rightarrow \text{infinite} \quad \La \sim 2-3 \text{ fm}
Lattice spacing \(a\) \rightarrow 0 \quad a \sim 0.1 \text{ fm}
(cutoff \(p/a \sim 6 \text{ GeV}\))

- \(\det(D_W)\) very expensive to include in \(Z\)
set \(\det(D_W) = 1\) \quad \text{quenched approximation}

ignore quark loops
• $D_W$ breaks chiral symmetry -- additive quark mass renormalization

heavy quarks ; $m_\pi > 350$ MeV need to extrapolate to small $m_\pi$

Around 1998 lattice fermion operators were discovered which maintain exactly chiral symmetry

Overlap and Domain Wall operators

Very CPU expensive – 30-50 times more than $D_W$
Nucleon Deformation

\[ S = \frac{1}{2} \]

Spectroscopic quadrupole moment vanishes, i.e. one photon measurements cannot reveal the shape

Intrinsic quadrupole moment w.r.t. body-fixed frame exists

\[ Q_0 = \int d\vec{r} \rho(\vec{r})(3z^2 - r^2) \]

\[ Q_0 > 0 \quad \text{prolate} \]
\[ Q_0 < 0 \quad \text{oblate} \]

modelling required!
$N \rightarrow \Delta(1232)$

\[ N(qqq) \]

\[ I = \frac{1}{2}, \ J = \frac{1}{2} \]

938 MeV

Spherical $\Rightarrow$ M1

Deformed $\Rightarrow$ M1, E2, C2

\[ \Delta(qqq) \]

\[ I = \frac{3}{2}, \ J = \frac{3}{2} \]

1232 MeV

Deformation signal

\[
\text{EMR} = \text{Re}\left( \frac{E_{1+}^{3/2}}{M_{1+}^{3/2}} \right)
\]

\[
\text{CMR} = \text{Re}\left( \frac{S_{1+}^{3/2}}{M_{1+}^{3/2}} \right)
\]
EMR & CMR
Experimental Status

uncertainties in modelling final state interactions

Thanks to N. Sparveris (Athens, IASA & JLab)
The Transition Matrix Element

\[ < \Delta(p', s') | J^\mu | N(p, s) > = i \sqrt{\frac{2}{3}} \left( \frac{m_{\Delta} m_N}{E_\Delta E_N} \right)^{1/2} \bar{u}_{\tau}(p', s') O^\tau \mu u(p, s) \]


\[ O^\tau \mu = G_M(q^2) K_M^\tau \mu + G_E(q^2) K_E^\tau \mu + G_C(q^2) K_C^\tau \mu \]

\[
\begin{align*}
\text{magnetic dipole} & \quad \uparrow \\
\text{electric quadrupole} & \quad \uparrow \\
\text{scalar quadrupole} & \quad \downarrow 
\end{align*}
\]

\[ \text{EMR} = - \frac{G_{E2}(q^2)}{G_{M1}(q^2)} \]

\[ \text{CMR} = - \frac{\vec{q}}{2M_\Delta} \frac{G_{C2}(q^2)}{G_{M1}(q^2)} \]
Lattice QCD Calculation

- generate a nucleon at $t=0$
- inject a photon with momentum $q$ at $t=t_1$
- annihilate a Delta at time $t=t_2$
- measure the 3-pt function
- extract the form factors from suitable ratios of 3-pt and 2-pt functions
\[ \langle G^\Delta_{ij}^{\mu N}(t_2,t_1; \vec{p}', \vec{p}; \Gamma) \rangle = \frac{\langle G^\Delta_{\sigma}(t_2, t_1; \vec{p}', \vec{p}; \Gamma) \rangle}{\sqrt{\langle G^\Delta_{ii}(t_2; \vec{p}'; \Gamma_4) \rangle \langle G^{NN}(2t_2 - 2t_1; \vec{p}; \Gamma_4) \rangle}} \]

\[ \langle G^\Delta_{\tau}(t_2, t_1; \vec{p}', \vec{p}; \Gamma) \rangle = \int d\vec{x}_2 e^{-i\vec{p}' \cdot \vec{x}_2} \int d\vec{x}_1 e^{-i(\vec{p}' - \vec{p}) \cdot \vec{x}_1} \langle \Omega | T(\chi_{\tau a}(x_2) j^\mu(x_1) \chi^N_b(0)) | \Omega \rangle \Gamma_{ba} \rightarrow Z_\Delta Z_N e^{-E_\Delta(t_2-t_1)} e^{-E_N t_1} [K_{M1}^{\tau \mu} G_{M1}(q^2) + K_{E2}^{\tau \mu} G_{E2}(q^2) + K_{C2}^{\tau \mu} G_{C2}(q^2)] \]

Known kinematical functions of $p, p'$

Select $\mu, \tau, \Gamma$ indices to isolate different form factors

\[ \Gamma_i = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_4 = \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \]
Lattice parameters

Wilson $N_F = 0$  \( \beta = 6.0 \)
- \( 32^3 \times 64 \)  \( a = 0.09 \) fm
- \( m_\pi = 0.56 \) GeV
- \( m_\pi = 0.49 \) GeV
- \( m_\pi = 0.41 \) GeV

Wilson $N_F = 2$  \( \beta = 5.6 \)  \( a = 0.08 \) fm
- \( 24^3 \times 40 \)  \( m_\pi = 0.69 \) GeV  \( \text{(TXL)} \)
- \( 24^3 \times 40 \)  \( m_\pi = 0.51 \) GeV  \( \text{(TXL)} \)
- \( 24^3 \times 32 \)  \( m_\pi = 0.38 \) GeV  \( \text{(DESY)} \)

Hybrid scheme
MILC $N_F = 2 + 1$ staggered (KS) sea
Domain Wall valence (\( L_5 = 16 \))
- \( a = 0.125 \) fm

<table>
<thead>
<tr>
<th>( a m_s )</th>
<th>( a m_u )</th>
<th>( m_\pi )</th>
<th>( L^3 \times T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.03</td>
<td>0.59 GeV</td>
<td>( 20^3 \times 64 )</td>
</tr>
<tr>
<td>0.05</td>
<td>0.02</td>
<td>0.50 GeV</td>
<td>( 20^3 \times 64 )</td>
</tr>
<tr>
<td>0.05</td>
<td>0.01</td>
<td>0.36 GeV</td>
<td>( 28^3 \times 64 )</td>
</tr>
</tbody>
</table>

Strange quark mass fixed on physical value.
quenched QCD results in
C. Alexandrou, Ph. de Forcrand, H. Neff, J. Negele, W. Schroers, A. Tsapalis
PRL 94, 021601 (2005)

dynamical QCD results in
C. Alexandrou, G. Koutsou, H. Neff, J. Negele, W. Schroers, A. Tsapalis
PRD 77, 085012 (2008)

dipole fits well – requires larger mass
1.3 GeV vs 0.78_{\text{exp}}

slower falloff at small $Q^2$

missing pion cloud

no unquenching effects visible
EMR

qualitative agreement with experiment
quadrupole amplitudes in $N \rightarrow \Delta$ verified from QCD

Low $Q^2$ data in agreement with chiral EFT predicted behavior

V. Pascalutsa & M. Vanderhaeghen, PRL 95(2005) 232001

deformed $N$ & $\Delta$ states
$q^2 = (p' - p)^2$

$$\gamma^* \Delta(p,s) \rightarrow \Delta(p',s')$$

$J = 3/2$

elastic matrix element can reveal quadrupole amplitudes

$$\langle \frac{3}{2} | Q | \frac{3}{2} \rangle \neq 0$$

$$< \Delta(p',s') | J_\mu | \Delta(p,s) > =$$

$$\bar{u}_\sigma(p,s) \left[ g_{\sigma\tau} \left( a_1(q^2) < J_\mu | J_\mu > + \frac{a_2(q^2)}{2m_\Delta} (p_\mu + p'_\mu) \right) + \frac{q_\sigma q_\tau}{4m_\Delta^2} \left( c_1(q^2) < J_\mu | J_\mu > + \frac{c_2(q^2)}{2m_\Delta} (p_\mu + p'_\mu) \right) \right] u_\tau(p,s)$$

four independent $q^2$ dependent form factors: $a_1, a_2, c_1, c_2$
reparameterize in terms of spin/parity based
physical form factors

\[
\begin{align*}
\begin{array}{l}
a_1(q^2) \\
a_2(q^2) \\
c_1(q^2) \\
c_2(q^2)
\end{array}
\end{align*}
\Rightarrow
\begin{align*}
\begin{array}{l}
G_{E0}(q^2) \\
G_{M1}(q^2) \\
G_{E2}(q^2) \\
G_{M3}(q^2)
\end{array}
\end{align*}
\]

Electric charge form factor
Magnetic dipole form factor
Electric quadrupole form factor
Magnetic octupole form factor

useful for the extraction of static properties & distributions

rms radius
\[
\langle r^2 \rangle = -6 \left. \frac{dG_{E0}(Q^2)}{dQ^2} \right|_{Q^2=0}
\]
magnetic moment
\[
\mu_{\Delta} = \frac{e}{2m_\Delta} G_{M1}(0)
\]
quadrupole moment (deformation)
\[
\sim G_{E2}(0)
\]
magnetic octupole moment
\[
O = \frac{e}{2m_\Delta^3} G_{M3}(0)
\]
Select $\sigma, \mu, \tau$ indices to isolate different form factors
Electric charge form factor & rms radius

\[ G_{E0}(Q^2) = \frac{1}{\left( 1 + \frac{Q^2}{\Lambda_{E0}^2} \right)^2} \]

rms radius from slope at \( Q^2 = 0 \)

\[ \left\langle r^2 \right\rangle = -6 \frac{dG_{E0}(Q^2)}{dQ^2} \bigg|_{Q^2=0} \]

- quenched Wilson, \( m_\pi = 410 \text{ MeV} \)
- hybrid, \( m_\pi = 353 \text{ MeV} \)
- dynamical Wilson, \( m_\pi = 384 \text{ MeV} \)

dipole fits well the data

\[ \sqrt{\left\langle r^2 \right\rangle} \text{ (fm)} \]

- quenched Wilson 0.652(9)
- \( N_F=2 \) Wilson 0.611(17)
- \( N_F=2+1 \) Hybrid 0.641(22)
Magnetic dipole form factor and magnetic moment

\[ G_{M1}(Q^2) = G_{M1} \exp \left( - \frac{Q^2}{\Lambda_{M1}^2} \right) \]

\[ \mu_{\Delta^+} \left( e / 2m_N \right) \]

- quenched Wilson: 1.811(69)
- \( N_F=2 \) Wilson: 1.58(11)
- \( N_F=2+1 \) Hybrid: 1.91(16)

chiral extrapolation

V. Pascalutsa & M. Vanderhaeghen,
PRL 95(2005) 232001

finite volume effect?
Electric quadrupole form factor & deformation

exponential fits with $Q^2$ to extract $G_{E2}(0)$

consistently negative $G_{E2}(0)$

Quadrupole moment $Q_{3/2}^\Delta$ is connected to transverse quark charge densities in the infinite momentum frame

.. which in turn are connected to the FFs at $Q^2=0$

$$Q_{3/2}^\Delta = \frac{e}{2m_{\Delta}^2} \left\{ 2 \left[ G_{M1}(0) - 3e_{\Delta} \right] + \left[ G_{E2}(0) + 3e_{\Delta} \right] \right\}$$

Consistently positive $Q_{3/2}^\Delta$ for spin $+3/2$ $\Delta^+$ state in the IMF

elongated along spin axis (prolate)
Conclusions

• Lattice QCD is the established technique for the study of hadron properties from first principles with controlled approximations.

• Hadron Structure can be reliably revealed through Form Factors, Structure functions and GPDs calculated on the Lattice.

• Nucleon deformation is studied through the $\gamma^*N \rightarrow \Delta$ transition form factors for $Q^2$ up to 1.5 GeV$^2$. EMR & CMR ratios in agreement to experiment and chiral EFT predictions for pion masses down to 350 MeV.

• $\Delta(1232)$ e.m. form factors are directly calculated on quenched and dynamical lattices. Magnetic moments, rms radii and quadrupole deformation are extracted with precision that supercedes experimental measurements.

• Similar calculations of axial & pseudoscalar form factors complete the picture of the complicated dynamics in hadron states.